Chapter 7
Reflections on History of Mathematics

History of Mathematics and Mathematics Education

Luis Radford

Abstract The specialization that mathematics education research has undergone in the past decades has led to a sense of division and disconnection between mathematicians and mathematics education researchers. This chapter deals with the possibilities that the history of mathematics may afford to reduce the divide. Although the recourse to the history of mathematics is an interesting prospect, it unavoidably induces new problems. A range of tensions becomes visible among the involved communities of teachers, historians of mathematics, mathematics education researchers, and mathematicians. Some of these tensions are investigated in this chapter, in particular in the case of a hermeneutic reading of original sources. The tensions that the history of mathematics induces, it is argued, may function as a way to foster a critical reflection and dialogue to contribute to a rich multi-layered understanding of mathematics, its history, and its teaching and learning.

Keywords History of mathematics · Hermeneutic approach · Teachers' beliefs about mathematics · Aesthetic in mathematical thinking · Mathematics and culture · Nature of mathematics

Introduction

This chapter presents a reflection on some of the contributions of the history of mathematics to mathematics education. It also explores the manner in which the history of mathematics may serve as a bridge across the intensifying divide between mathematics and mathematics education research. While the first aforementioned point has been a matter of extensive discussion (see, e.g., Barbin et al. 2008), the

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second point results from the increasing specialization of mathematics education as a research discipline. It is clear indeed that in the early 20th century, research in mathematics education revolved around curricular problems and international cooperation, as the epoch-making articles published in *L'Enseignement Mathématique* in the first decades of the 20th century make clear (see, e.g., Borel 1914; Bourlet 1910). During the 1970s and 1980s, research in mathematics education moved to new arenas: problems of a psychological nature moved to center stage, with an interest in understanding the students' thinking, and, more recently, a shift has become clear with the political concerns of today (see, e.g., the Educational Studies in Mathematics Special Issue edited by Brown and Walshaw 2012), where a need to go beyond the definition of mathematics education as the diffusion of the mathematical content (see, e.g., Brousseau 1997) is questioned, if not contested. Often, these shifts have led to a sense of disconnection between the work of professional mathematicians and mathematics education research. In the latter, new terminologies, concepts, and methods have been introduced and developed, often with the recourse to theories in the social sciences, such as linguistics, semiotics, sociology, and anthropology. Sometimes these new trends, which are clearly embedded within social science research, appear as excessive and even unnecessary (Eisenberg and Fried 2009).

The chapter has its origin in a symposium organized in Beer-Sheva in 2012 to honour the seminal work of Ted Eisenberg. The symposium was an extraordinary opportunity to reflect on these matters and to try to come up with possible actions to overcome the separation that seems to affect the communities of mathematicians and mathematics education researchers. By focusing on the history of mathematics, the contributors to this chapter present some of the potential and challenges that such an endeavour entails. The following sections still retain the aural dimension of the presentations made during the Symposium. We decided to keep them this way for reasons that will become apparent later.

In the opening section, Alain Bernard reflects on the challenges that a hermeneutic approach may present to the teachers of mathematics. An interpretative turn to historical texts, Bernard argues, entails some skills and knowledge that teachers may be lacking in order to ensure suitable hermeneutic classroom discussions. Michael Fried stresses the tension that a historical attitude induces in mathematics, mathematics teaching, and mathematics education research. He suggests that the recourse to history in mathematics education may bring a philosophical view that may provide historians and mathematicians with an opportunity for even philosophical discussion and historical understanding. Fulvia Furinghetti argues that the history of mathematics offers a unique window through which teachers' beliefs about mathematics can be made explicit and turned into possibilities of conceptual growth and development. Nathalie Sinclair puts forward an interesting conception of mathematics as temporal and material activity embodying diverse modes of thought and forms of subjectivity. In addition to exploring the past through the written dimension of historical texts, a hermeneutic approach could also encompass mathematics as performance and unveil the richness of mathematical narrative styles. In my own section, I ask three questions and make a remark. They intersect with Sinclair's arguments. The questions are a rhetorical device used to invite mathematicians and mathematics educators to rethink what we mean by mathematics. The thrust of the questions is my concern with the fact that mathematics has unfortunately become a technical domain under the influence of contemporary neo-liberal forms of production and its emphasis on marketing and consumption as the modern and postmodern predominant forms of life.

History Within Math and Science Teaching: A Historical Issue

Alain Bernard

My reaction to Jahnke's insightful proposal for a *hermeneutic* approach of the history of mathematics in mathematics teaching is guided by a few basic claims that are indicated in my title. The first is that there are difficult questions underlying the introduction or promotion of history of mathematics into math teaching. The second is that we are now in a situation in which the history of mathematics has come to be considered almost as a *necessary* component of mathematics education in many (though not all) countries. Finally my central claim is that reflecting on the above questions should be deeply informed and guided by a *historical* reflection about our current situation.

I do think, first of all, that "mathematical education" should be understood just in the general way that was underlined by Presmeg: namely as *the activity of teaching mathematics* or even, as was suggested by Niss' categories, as *the organization of the human and institutional framework for it*. By contrast, both Presmeg and Eisenberg have rightly reminded us that math education *research* is a much more recent academic field that did not exist at the beginning of the 20th century or before. From a historical perspective, recognizing these facts is crucial, for it enables one to say, without anachronism, that several major (and also lesser known) mathematicians have been constantly and deeply involved in math education, that is, into the reform of math curricula and pedagogical methods *from the 19th century to the present day*. For it must be recalled that the *level of interest* taken in math education by mathematicians in particular or by political people in general *has neither really decreased nor changed*. As Niss shows very clearly, there are still active mathematicians, as well as many other people, that pay much interest to mathematics education and try to influence it in deep and significant ways. Following Jahnke's reaction to Niss' talk, one should add the timeline to his picture: in other words, what is true today has already been true for a long period.

There are, in turn, deep and long lasting historical reasons, for which many people have been concerned by (if not involved in) the *constant reform* of math education since the 19th century. My remarks will elaborate on one important point made by Movshovitz-Hadar about the *ever evolving perimeter of mathematics* and its necessary consequences on mathematics education. Indeed, a recurrent argument seems to evoke a golden age in which the very notion of what math education is about was clearly defined and not a subject of contention. But I fear the Golden Age was already an Iron age: the eve of the 20th century was already characterized by dramatic and significant changes in orientation and subject matter. Thus, introducing the concept of function as a fully legitimate subject for mathematics education was precisely the move promoted by mathematicians like Borel, Klein or Poincaré, who were deeply involved in the reform of curricula in their respective countries: before then, the concept of function was not a central subject in math education, but more a subject for advanced studies.

Many other examples could be given. But the question is: why would those people have felt the urge for change? Generally speaking, we have to remind ourselves that mathematics itself, throughout the 19th and 20th century, and especially after WWII, has constantly undergone drastic and profound changes—more than in every other period. These changes have been quite directly related to the changes in industrial societies: new technologies, new industries, new sciences. These rapid and radical changes have naturally led many people, first of all mathematicians, to consider the unavoidable fact that questions concerning mathematics and their education on a large scale could *not* be any more considered in isolation of many others. Those other questions, which are so constantly present in the background, such that we tend to forget them, touch on the development of industry, experimental science, economy or politics—especially educational policies. For example, much of the concern about mathematics education was fostered by the launch of Sputnik in 1957 and by the kind of *shock* it produced in western countries, and first of all in the USA, by that time. To my view, this remark can and should be extended to much of the history of the 19th and 20th century mathematics and mathematics education, with an acceleration after WWII.

I thus come to the last historical fact that I would like to connect to the previous ones. Ever since there has been a question of changing (mathematics) education to accommodate the 'new' industrial world and its needs, namely from the 19th century onwards, *it has also been a question of introducing history of science in general (history of mathematics in particular) into science curricula*. This was explicitly done, at least in France in which this contextual question has been much studied, in order to *compensate* the inevitable split induced by the twofold curricula, separating science from literature tracks—see, among others, Hulin's (2011) synthesis. Very early, during the industrial age, therefore, a teaching of history of science was welcomed and called for—at least theoretically.

The situation of today has made this traditional wish all the more central than it has largely entered official curricula: the teaching of science in general, and of mathematics in particular, is now meant to be deeply 'cultural': this means, in particular, that it should include more history, more epistemology and more facts concerning society at large. We should be strongly aware of the fact that this is *not* a recent move, but already the continuation of long-lasting concern; and that this concern, in turn, should not be dissociated from the history of modern science, industry and philosophy (especially positivism). We should finally be aware of the practical implication of these moves: namely that science and mathematics teachers are requested, more than ever, to deal with science and mathematics as cultural subjects, however this might be interpreted.

With these historical observations and preliminaries in mind, I think we can address Jahnke's proposals and questions. First of all, I can only agree with Jahnke's suggestion that readers of historical sources of mathematics should be placed in the situation of *interpreting* these sources, and that *reading* should be considered a hermeneutical task in a strong sense. Indeed, this proposal addresses somewhat in depth the real difficulties encountered by teachers when they try to address the long-lasting demand signaled above.[1] But such proposals should be understood against the historical background that I have summarized before. For, to put it simply, we retrieve the bizarre situation I have spoken about:

Most students of mathematics who are now asked, as teachers or future teachers, to develop the kind of pedagogical activities that are indispensable from interpretation, are not ready for it. From my experience, many such teachers were often not prepared to organize a discussion on a complex argument, implying the elaboration of a coherent interpretation that might be expressed orally or by written means.[2]

Teachers of literature, philosophy or history, by contrast, would consider natural that students should develop a complex point of view on a given document; this idea is more or less alien to many students in science.

The split is not only a problem of being trained or not in the kind of pedagogical activities that foster interpretation, but also, of course, a question of epistemology of what science and literature studies are about: there is a kind of invisible limit here that forbids such cooperation. We are collectively far away from the old idea that mathematics and science are a legitimate part of literature in a strong sense.

On the other hand, we are perhaps not so far away from these classical ideas. For these questions and implicit frontiers are now quickly changing: the new technologies and especially the rise of "digital humanities" are deeply changing the way we read and write. This does not mean that printed matter is outdated, but we must nevertheless count on the new media and on the way in which they radically transform the access to historical and cultural information, as well as the way students and teachers might cooperate with each other. This recent change should probably be part of the discussion.

[1]Other proposals, like Michael Fried's notion of a "radical accommodation" between history of mathematics and mathematics teaching seem to come close to it—although I understood from Michael himself that Jahnke's proposal is not yet radical enough to correspond to the aforesaid category.

[2]For example, I am ready to bet that the kind of "open discussion" that Jahnke organized with his students on Bernoullis' conception of infinitesimals quantities would be dreadful to many mathematics teachers: both the spirit and the concrete organization of such debates is in many cases alien to them.

Mathematicians, Historians of Mathematics, Mathematics Teachers, and Mathematics Education Researchers: The Tense but Ineluctable Relations of Four Communities

Michael N. Fried

Rather than asking whether history of mathematics is good or bad for students or for teachers of mathematics—and I *do* think it is good!—I would like to focus on how our presuppositions about mathematics and about the history of mathematics play out in the relations among four communities, that is, to the extent each is concerned with history of mathematics in mathematics education. I have in mind mathematicians, historians of mathematics, mathematics teachers, and mathematics education researchers. These different communities cannot be assumed to speak in one voice. I would like to suggest that mathematics education as a whole must somehow situate itself within a web of tensions created by the interests and commitments of these four communities. Furthermore, I would suggest that mathematics education research, though it itself is a pole within this web, has a distinct role of creating a view of mathematics education in which these tensions can be productive for mathematics learning, not paralyzing.

To begin, I should remark that the relationship between the *history of mathematics* and mathematics education mirrors the overall problem of this book namely, *mathematics* and mathematics education. And continuing down this hall of mirrors, the relationship between mathematics and history of mathematics themselves reflects the problem as well. As with mathematics educators, historians of mathematics until the middle of the twentieth century were invariably well-trained mathematicians, most often working mathematicians. The awakening of history of mathematics as an independent historical discipline created a certain amount of tension within the mathematics community just as the crystallization of mathematics education as a separate academic discipline has. When, for example, Unguru wrote that the history of Greek mathematics needed to be revised (Unguru 1975) so that it would be based on sound history rather than sound (modern) mathematics, he was attacked by mathematicians as not understanding mathematical thinking, as he was attacking them for not understanding historiography (e.g. van der Waerden 1976).

Taking into account history of mathematics in mathematics education adds another level of complexity, however (it was this that was analyzed in Fried 2001, 2007). For history of mathematics as *history* tries to see the how mathematics of the past was *different* from the mathematics of today. With that, it treats mathematics as a product of culture and, as Judith Grabiner (1974) has put it, as "time dependent." This means one cannot assume a modern mathematician should be the final arbiter in judging what was said or done in the mathematical past. To the extent then that mathematics educators turn to a cultural view of mathematics, they align themselves with the history community. On the other hand, because working mathematics teachers must teach mathematics with an eye to its application in the sciences or its investigation in mathematics itself, it must give mathematics an unconditional objectivity that aligns them with the mathematics community or, more generally,

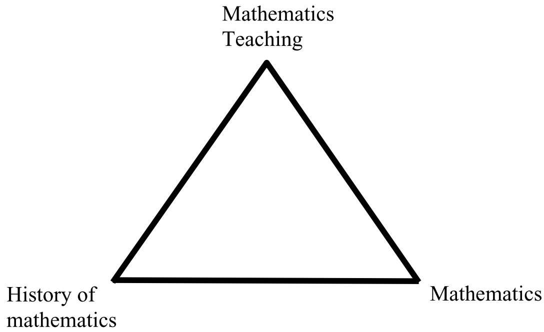

Fig. 7.1 A three-way tension between mathematics, mathematics teaching, and the history of mathematics

with those who see mathematics of the past as fundamentally the same as the mathematics of today. Pulled in this latter direction, educators are hardly obliged to take history into account in their teaching; that is, history can be added or subtracted at pleasure. I would point out that even Freudenthal, who supported the inclusion of history in mathematics education vigorously, still did not think history could help one understand better the subject matter of mathematics (Freudenthal 1981).

So there arises, to start, a kind of three-way tension (Fig. 7.1):

(1) Between mathematicians for whom mathematics exists outside of time waiting to be discovered, and historians of mathematics who see mathematics as a cultural product which by definition develops in time;
(2) Between mathematicians concerned with mathematical content and ideas, and mathematics educators concerned with the development of students' mathematical thinking and their situating their mathematical understanding within their culture and everyday experience;
(3) Between mathematical educators concerned with preparing students for work in science and engineering and the concomitant need to teach them modern mathematical procedures and concepts, and historians of mathematics who keep the present at a distance in order to understand the past and who we are as beings possessing that past.

But despite the symmetry of the triangle, there is in fact a clear asymmetry between the mathematicians and historians of mathematics on the one hand and the mathematics educators on the other. For while mathematicians and historians of mathematics (taken as types, of course, not individuals) are fairly consistent in their respective positions, mathematics educators are somewhat chameleon-like in theirs. This, however, is the nature of the subject: mathematics education has a diverse set of ends that are themselves not completely consistent. It is for that reason that mathematics education can by itself mirror the tension between mathematics as a discipline and the history of mathematics.

The fourth pole is mathematics education research, which Norma Presmeg was careful to distinguish from mathematics teaching in Chap. 4. And with this fourth pole, we have, in fact, a tetrahedral web of relations. This is represented in the tetrahedron below (Fig. 7.2), where each face corresponds to a different simplicial set

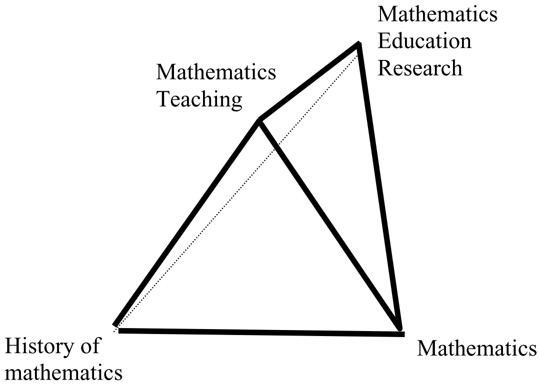

Fig. 7.2 A four-way tension between mathematics, mathematics teaching, mathematics education research, and the history of mathematics

of tensions (of course the face whose vertices are mathematics education research, mathematics, and mathematics teaching, respectively, is that which is the main focus of this book).

The relationship of mathematics education research to history of mathematics and mathematics is in many ways similar to that of the mathematics teacher: the disciplines of mathematics and history of mathematics make conflicting demands as to what the teacher is supposed to teach, as well as to what the researcher is supposed to investigate. Does mathematics education research waste time if it does not try to find ways of teaching functions or how to solve a system of linear equations? Does it neglect the "true hard-core" of the subject, if it looks at mathematics as a semiotic-cultural system?

However, mathematics education research, as an academic community, has considerably more freedom to define itself and shape a view of mathematics than does the community of mathematics teachers. Mathematics education research possesses, accordingly, a greater potential to bridge the divide between mathematics and history of mathematics by defining a view of mathematics education which can accommodate both. As I emphasized in Chap. 2, this means defining what exactly it is to be mathematically educated. Mathematics education research may take on an important function in enriching mathematics itself in this regard, while also defining a role for mathematics teachers in such a way that situates them neatly between mathematics and history of mathematics. A tetrahedron such as that in Fig. 7.3 might, therefore, better show the relation between mathematics education research and the communities of teachers, mathematicians, and historians, namely, that of a kind of orchestrator poised above the plane containing mathematics teaching, history of mathematics, and disciplinary mathematics.

In accepting this bridging role, both with regards to mathematics and history of mathematics and between mathematics teaching and both of these disciplines, mathematics education research must contend with questions both of a practical and theoretical nature.

On the theoretical side, we ought to ask questions such as the following: Does the introduction of history of mathematics into mathematics education, where this is not a trivial introduction for the sake of motivation or "spicing up" mathematics lessons,

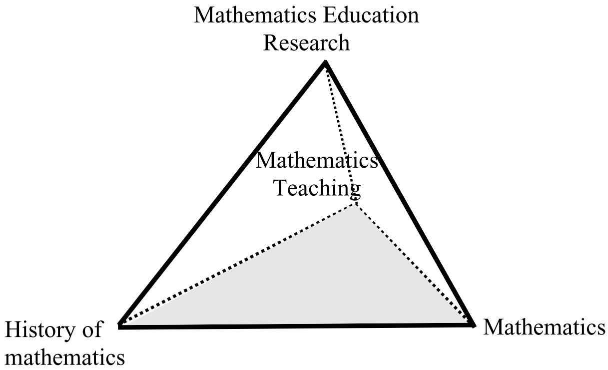

Fig. 7.3 The relation between mathematics education research and the communities of teachers, mathematicians, and historians

imply a philosophical position with respect to mathematics? Does this position distance mathematicians from mathematics education, for example, by weakening the claim that modern mathematical understanding is the key to historical understanding? Can mathematics education act as a context for a philosophical discussion of mathematics where mathematicians and historians can have an equal say (a kind of amplification of the classroom setting Lakatos chose for his historico-philosophical discussion of proof)?

On the practical side, I would suggest that bringing history of mathematics into mathematics education in such a way that it is both mathematics and also truly history of mathematics consists, first of all, in the study of original texts—really looking at Euclid's *Elements* or working through Descartes' *Geometrie*. How to introduce original texts, however, is not obvious. In its most uncompromising form, learning by way of original texts falls under the category of what I called in the past (Fried 2001), "radical accommodation" (as opposed to the other radical alternative, "radical separation"). It is radical because the texts become *primary in every sense of the word*: the study of mathematics in "radical accommodation" becomes precisely the study of mathematical texts, just as literature is the study of great works of prose and poetry. I had claimed that this would indeed also be mathematics:

> ...in the study of mathematical texts, one is not only engaged in solving problems and developing ideas with a great mathematician, and therefore becoming deeply acquainted with the human activity of mathematical work, but one is also engaged in a kind of reflective thinking or inquiry that ultimately is of the highest importance for one who deals with *technical* scientific and mathematical work. (Fried 2001, p. 402)

But original texts can be introduced without making them the exclusive source of learning. Indeed, one can gain much historical understanding by bringing mathematics classroom learning to original texts *if* one is made ever cognizant that one is indeed an interpreter with a point of view. This is the core of what Jahnke calls the "hermeneutic" approach (e.g. Jahnke 2000). In taking up a theme such as the hermeneutic approach, mathematics education research provides an example of how it can investigate a practical approach to learning that defines a set of mutual relationships between students and teachers of mathematics, modern mathematical knowledge, and historical understanding.

History in the Mathematics Classroom

Fulvia Furinghetti

One of the struggles inside the community of HPM (International Study Group on the Relations between History and Pedagogy of Mathematics, affiliated to ICMI) is to make clear the relation between history and pedagogy of mathematics. Some papers written by Ted Eisenberg provide hints for pointing out the links between these two domains. In the following I will refer in particular to teachers' beliefs.

Let me start with the claim by Eisenberg (1977) that there is close to a zero correlation between teacher knowledge and student achievement, and that other factors appear to be responsible for student achievement. Among the factors responsible of this failure in teaching I consider important teachers' beliefs about mathematics and its teaching. These beliefs, for example, make teachers neglect what they learnt at university and reproduce in their classroom what they have been taught in secondary school. I am not the first to take this point of view. At the beginning of the twentieth century two important mathematicians engaged in mathematics education and focused on the problem of teachers' knowledge by pointing out this fact (Borel 1907; Klein 1924).

Then, a main aim of teacher educators is to challenge prospective teachers' beliefs. As I have discussed in some papers, history may be a good tool for attaining this aim, since it provides an unknown landscape where people are obliged to look at things from a different perspective and to grasp aspects that previously escaped their attention (Furinghetti 2007). As an example, I mention the case of algebra. Usually teachers tend to consider algebra as an extension of arithmetic, generalization, abstraction, and use of symbolization. I challenge this view by asking them to solve medieval problems such as the following Problem 47 taken from *Trattato d'Aritmetica* by Paolo Dell'Abbaco:

> A gentleman asked his servant to bring him seven apples from the garden. He said: "You will meet three doorkeepers and each of them will ask you for half of all apples plus two taken from the remaining apples." How many apples must the servant pick if he wishes to have seven apples left?

In solving this kind of problem two paths may be followed, which may be put in relation with the analytic and synthetic methods:

– arithmetic path: from the known (left apples) to the unknown (apples to be picked)
– algebraic path: from the unknown (apples to be picked) to the known (left apples)

Reflecting on these paths leads the teachers to focus on the fact that algebra is not only generalization, not only abstraction, not only using symbols, not only an extension of arithmetic: algebra is a method and the analytic method is its core. Then François Viète's introduction of parameters and variables is not perceived as something coming out of the blue, but as a consequence of this way of looking at algebra.

A further belief challenged by history is the view of the role of intuition and rigor in mathematics teaching and learning (Dreyfus and Eisenberg 1982). Mathematicians such as Poincaré (1899) and Klein (1896) considered the history of mathematics a suitable context for bringing intuition back into the teaching process against the excesses of rigor advocated by some of their contemporary colleagues.

The main idea expressed by Klein is that students need to approach a topic at an "intuitive" level and later on to pass to the formal level. History may be useful in this regard because it brings back the polished concepts as are presented in the modern textbooks to their origin. History recovers the cognitive roots, described by Tall (2003) as concepts which are (potentially) meaningful to the student at the time, yet contain the seeds of cognitive expansion to formal definitions and later theoretical development. The historian Gino Loria, who was a convinced supporter of the use of history in mathematics teaching, epitomized this idea about cognitive roots by using a sentence found in Victor Hugo's novel *Les travailleurs de la mer*, which says that any embryo of sciences presents this double aspect: monster as a fetus; marvel as a germ (Loria 1914, p. vii).

The focus on formal approaches has the consequence that a vast majority of students do not like thinking in terms of pictures (Eisenberg and Dreyfus 1991). This way of thinking may be promoted by history, since the early stages in the development of concepts often reside in the visual domain. This aspect has been exploited in significant experiments concerning the teaching of calculus.

Another challenge to teachers' beliefs is to make them shift the focus of their teaching from product to process. This shift may be fostered by history, since reading original sources directs the attention to processes, which leads to the genesis of concepts. The engagement promoted by the contact with an author's thinking obtained by means of historical passages is an aesthetic value introduced into mathematics teaching. In the words of Hawkins (as cited in Featherstone and Featherstone 2002), aesthetics "is a mode of behavior in which the distinction between ends and means collapses; it is its own end and it is its own reinforcement" (p. 25). Then history of mathematics may be a means for introducing a form of aesthetics into the mathematical discourse in the classroom, as advocated by Dreyfus and Eisenberg (1986).

The few aspects outlined above suggest ways in which the history of mathematics challenges teachers' imagination in finding new modes of dealing with mathematical discourse in the classroom.

Aesthetic Considerations

Nathalie Sinclair

This chapter offers some reactions to Jahnke's chapter, which outlines the possibility of using historical texts in the mathematics classroom to enable students to (1) develop insights into the development of mathematics, (2) develop an understanding of the role of mathematics in our society and (3) encourage the perception of the subjective dimension of mathematics.

I would like to contribute to this discussion by way of Eisenberg's 1986 paper (with Dreyfus) on the role of the aesthetic in mathematical thinking, which has had a strong influence on my own work in mathematics education. They suggested that explicit attention to aesthetics in the mathematics classroom could help improve students' problem-solving abilities. One challenge that mathematics educators must reckon with is not only finding ways of welcoming—and even eliciting—aesthetic values in the classroom, but also accepting that the values students have in the classroom, with respect to mathematics, do not always align with those of mathematicians (see Sinclair 2001).

Of course, judgments of aesthetic values are not only subjective, but also strongly influenced by socio-historical factors. Yes, it is true that adjectives such as symmetry, order and precision re-occur across different historical time periods and in diverse cultures. But these are very broad descriptors that do not just operate in mathematics. Moreover, there are many examples in the history of mathematics where asymmetry, chaos and fuzziness also vied as aesthetic values.

So, one way of thinking about the historical project proposed by Jahnke is to focus not only on the changes in mathematical content, but also changes in the aesthetic values that constitute the discipline. These values become evident when one looks, for example, at the dominant activities of the day—perhaps focusing on problems related to the foundations of mathematics or on solving specific open problems. They also become evident when considering the techniques and strategies that are used to solve these problems—be they algebraic means or experimental ones. Finally, they become evident when questioning the reasons for focusing on certain problems or techniques over others—because they are more beautiful, right, useful, ideal or true. In other words, aesthetic considerations of the historical variety would concern *what* was attended to, *how* it was attended to and *why* it was attended to. These questions could easily be raised in the context of historical activity in the classroom and the answers, I think, would certainly support the threefold aims of Jahnke's proposal.

From an aesthetic point of view, much can be learned about mathematics and the people doing mathematics by reading historical texts. One can focus, for example, on the styles of writing that are used. Netz (2009) has argued that the Archimedean so-called "ludic" style of writing, which he characterises as involving narrative surprise, mosaic structure and generic experiment, and a certain "carnivalesque" atmosphere, evokes very different aesthetic qualities than, for example, the Euclidean style. Archimedes writes mathematics to delight and inspire; Euclid does so to organise and convince through strictly logical means. Clearly, these considerations are strongly related to the role of mathematics in society (to entertain? To be useful? To be pure?). It also suggests that the issue of subjectivity is not just of epistemological interest, but also of aesthetic order.

I have been arguing that the study of historical texts can (and should) have a strong aesthetic component to it. But now I'd like to raise questions about the limitations of this approach, still within the context of aesthetics. I will discuss three concerns:

The Historical Text Can Reveal Only the "Body" of Mathematics The first relates to the historian of mathematics Corry's (2006) distinction between the *body* and the *image* of mathematics. In contrast to the body of mathematics, which includes "questions directly related to the subject matter of any given mathematical discipline: theorems, proofs, techniques, open problem," the images of mathematics "refer to, and help elucidating, questions arising from the body of knowledge but which in general are not part of, and cannot be settled within, the body of knowledge itself".

Thus, while the body of mathematics might concern itself with describing a technique used in the course of a proof, the images of mathematics refer to the motivations, choices and values related to the use of certain techniques. While the body of mathematics concerns itself with defining objects, the image of mathematics questions which objects are defined and which are not. As Corry points out, mathematicians do not customarily write about their images. But images of mathematics constitute a layer of mathematical knowledge, one that centrally involves aesthetic concerns—and one that will not easily be revealed in a study of historical texts. Indeed, Netz's study of Archimedes' style required the use of sophisticated and specialised analytic tools from disciplines such as archaeology and cognitive linguistics. I would argue that it is just as much, if not more, in the changing images of mathematics, that we can learn about the development of mathematics, the role of mathematics in our society and the subjective dimension of mathematics.

From Written Text to Performance A second point I would like to raise relates to the idea of mathematical writing style that I have already mentioned and the question of what kinds of historical texts we might choose for students to read. Consider some of the linguistic features of modern mathematical writing that attempt to render utterly transparent the 'logical structure' of the text. These include the prevalence of non-active verb forms, the lack of direct address and the frequent use of imperatives. One can also read a more covert agenda aimed at creating the very sense of decontextualised authority and certainty that is then claimed as the hallmark of mathematics (Pimm and Sinclair 2009).

In any case, Solomon and O'Neill (1998) have usefully identified two contrasting styles of writing using a variety of texts authored by the 19th-century mathematician William Rowan Hamilton (the narrative and the paradigmatic). They argued that the main difference between these contrasting styles lies precisely in this 'glue' of logical versus chronological structuring (and their surface manifestations in terms of verb tense, personal pronoun use, connectives between sentences and other lexical choices). Interestingly, in Hamilton's range of mathematical writing, the syntactic glue changes depending on whether he was writing diary notes to himself, letters to friends or journal articles or monographs (ostensibly addressed to his colleagues). Of course, in all the writing, quaternions are the central topic. But when choosing a text for students to read, we might ask which piece of writing would work better in the classroom? Might the diary notes provide greater scope for writing about images of mathematics? Might studying the transition from the diary notes to the journal article help make explicit the ways in which professional mathematical communication seeks to immanent, immaterial truth and obscures personal motivations, feelings and doubts?

Because published texts tend to be the endpoint of mathematical investigation, both of the problem-solving process and of the writing process, these texts give a limited sense of mathematical activity. They fail to convey the narrative modes of thought that characterize discovery and, hence, run the risk of distorting the development of mathematics and even maintaining its objectivity. At issue, I think, at least in part, is the technology of the written word. As Brian Rotman (2008) has argued, the sequential logic of the printed (and copied) word in and of itself, independent of style, has a character of immanence and immutability.

It is interesting to imagine alternatives to the text in the historical project that Jahnke proposes. Consider, for example, being able to study the live performance of Archimedes, drawing geometric figures in the sand, going back and forth from diagram to symbol to gesture to spoken word. Or a YouTube video of a Terrence Tao lecture. The perception of subjectivity would be inescapable. Mathematics would be a temporal, material activity. In our digital era, not only is performance gaining ground over textual forms of communication, but the ability to manipulate time (reverse it, repeat it, fast forward it) will change the mathematical discourse. I wonder how classroom activities centered on historical texts will have the effect of celebrating a static, alphabetic way of mathematical communication.

Whither Subjectivity, Agency and Materiality? My third point relates to this discussion of the authority of the written word. It stems from the ideas of the historical and philosopher of mathematics Gilles Châtelet (2000) whose interest laid in the subjectivity, materiality and embodiment of mathematics. He studied several inventive instances in the history of mathematics, such as Hamilton's quaternions, Grassman's theory of the extension and Cauchy's residue theorem. But he studied these examples by analysing the diagrams that these mathematicians used to create new objects and relationships. For Châtelet, diagrams transduce the mobility of the body; they are "concerned with experience and reveal themselves capable of appropriating and conveying 'all this talking with the hand'."

And thus, his analysis of these historical episodes is an analysis of the these two, intertwined pivotal sources of mathematical meaning, mutually presupposing each other, and sharing a similar mobility and potentiality. Diagramming and gesturing are embodied acts that constitute new relationships between the person doing the mathematics and the material world. For Châtelet, the study of mathematical texts is not just an epistemological undertaking but an ontological one—the points and lines in the diagram do not represent ways of thinking about mathematical objects or spaces; rather, they *are* those objects and spaces; they can move, extend, cut, meet.

Although his study is an historical one, Châtelet's aims are philosophical. They challenge received notions of mathematics, insisting on its materiality, seeking to close the gap Aristotle erected between the abstract immobile mathematics and the concrete, mobile physical. But in terms of this book section, and of Jahnke's proposal, many questions come to mind: At the most general level, and similar to my previous point, does the study of text run the risk of ignoring an important part of the development and subjectivity of mathematics? At a more specific level, might the study of texts also include a Châtelet-like study of diagrams, not so much for its philosophical implications, but as a way to excavate the embodied meanings that created the objects and relationships under study? Lastly, is there room not only for Jahnke's epistemological laboratory, but also an ontological one in the mathematics classroom?

Three Provocative Questions and One Remark

Luis Radford

I start with a general observation. When we try to convince people of the benefits of history in mathematics education we resort to several possibilities—for instance, that the history of mathematics may help our students to attain a better understanding of the mathematics that they are learning today or to make the students sensitive to the fact that mathematics is a cultural construction.

Although laudable, our reasons tend to leave some views unquestioned. We tend to talk as if there were *one* mathematics, *one* history, and *one* history of mathematics. Perhaps we should start by asking ourselves what we mean by mathematics; only then might we be able to deal with the question of its possible histories.

Rationalist epistemologies present us with a view according to which mathematics is a body of objective knowledge that predicates truths that were already true even before they were discovered. If this is so, what then is the role of culture in the construction of mathematics? Culture, it turns out in rationalist accounts, is something that can only constrain or accelerate the rhythm of mathematics evolution but can in no way modify its natural course.

Yet, studies such as those of Emmanuel Lizcano (2009) bring to light the fact that mathematics is immersed in cultural and historical symbolic systems on which it draws its basic concepts, like those of number and figure. These cultural symbolic systems function as a semiotic superstructure that endows with meaning mathematical ideas and activities. In the case of ancient Greece, the whole mathematical edifice was governed by epistemic and ontological beliefs organized around the distinction between Being and Non-Being, and the logical principle of the Excluded Third. In the case of ancient Chinese mathematics, by contrast, mathematical thinking was organized around the *yin-yang* opposition. Since within the oppositional context of the yin-yang ontology each number has to have an opposed counterpart, what we now call negative numbers were "natural" to Chinese mathematics and remained unthinkable to the Greek *episteme*. To come up with the idea of negative numbers in Western culture, it was necessary to wait for the creation of new forms of labour and production and in fact to invent capitalism and its mercantilist practices of debt. (This does not mean that debts did not exist before. They did, but not in the typically surplus capitalist sense.)[3]

This short example opens up a possibility to try to envision, in new non-rationalist terms, the question of the nature of mathematics and its relationship to culture (Radford 2008). Of course, this example is not an isolated one. Current research in ethnomathematics offers a multitude of examples of mathematics that are quite different from the one we grew up into—many of them practiced orally only, as the Pythagorean brotherhood did in its own time.

The Provocative Question Is: Should We Be Concerned with Those Mathematics? I am not referring only to mathematics in other cultures that have made substantial contributions to our mathematics (e.g., Arabic mathematics and its impressive development of algebra). What I have in mind is the mathematics of cultural formations such as the one of the Lobodan people of the Normanby Island in Papua New Guinea. Lobodan mathematics is very distinctive in that it remains a-numerical. Lobodan people think relationally in ways that are different from ours. Drawing on an epistemology that is different from mainstream Western epistemology, Lobodan people do not quantify as we do: they compare in contextual *ad hoc* ways (Radford 2008).

Art scholars seem to be more prone to navigate between cultural forms of art than mathematicians are to navigate between radically distinct cultural forms of mathematical thinking. The goodness of concerning oneself with other mathematics—mathematics of other ethnical formations, present and past—bears on the question—I think I can already hear it—of *why*? Why should we be concerned with the mathematics of other cultural formations?

From a utilitarian viewpoint perhaps there is no reason. The mathematics that has been developed in the West is precisely the one that responds the best to the

[3]One of the oldest examples of negative numbers in the Renaissance appears in Nicolas Chuquet's Triparty en la Science des Nombres (Marre (ed.) 1880). Chuquet tackles a problem dealing with a merchant who has bought two kinds of cloths of different price. The total amount of pieces of cloth and the total amount of money are known. Solving what we would now call a system of linear equations, Chuquet finds out that the amount of pieces of cloth of the first kind is 15; he infers that the amount of pieces of cloth of the second kind is equal to 15 minus $17\frac{1}{2}$ and concludes that the problem is impossible, unless one interprets the difference $(-2\frac{1}{2})$ as a debt: the merchant bought $2\frac{1}{2}$ of cloth on credit! ("creance"; see Spiesser 2006, p. 19). In the following centuries, when algebraists like Bombelli and others do calculations on negative numbers, they are drawing on a conceptualization that has its roots in a commercial practice that has offered the possibility to think of negative numbers in a specific way—a social practice that has provided algebraists like Bombelli and Cardano with the conceptual ground to carry out a formidable cultural abstraction.

needs of progress as it came to be understood in the early 19th century—progress in a technological sense, where mathematics became the right hand of massive industrialization. But this is precisely my point. We need to rethink the *nature of mathematics*. Is mathematics technical stuff only? I think that most mathematicians would agree that the answer is no. In fact, there is a long list of philosophers (among them Hegel and Heidegger) who perceived the danger of reducing mathematics to its technical aspect, to a science of computation, to a kind of sophisticated technology. This conception of mathematics, the philosophers argued, eradicates the individuals from the discipline. Their concern was the depreciation of the subject of historical-cultural action. The student of mathematics is put on one side; mathematical knowledge is put on the other. Their contact is in the technological point. No wonder that mathematics is often found to be an unappealing subject by so many.

This discussion brings me to my second two-fold question: If mathematics is not technical stuff only, what else is it? And how can we take into account this neglected albeit important dimension of mathematics in school mathematics?

We need to rethink the nature of mathematics in general and the nature of school mathematics in particular. The technification of modern societies from the 19th century on led to a technification of mathematics. The justification of mathematics shifted from a discipline dealing with truth to one dealing with the efficient mastering of nature and the search for an optimal mechanism of production (Radford 2004). In the course of this process truth became obsolete. Euclidean geometry, to give but one example, has now disappeared from many school curricula. And if some vestiges can still be noticed, they are the remnants of the past. In Ontario, where I come from, what remains of geometry is what is susceptible to be translated into calculations. Our students do not prove theorems. They calculate. We have analytic geometry now.

My aim here is not to plead for a return to Euclid—at least not with the idea of resurrecting the splendors of truth as the Greeks conceived it. I take an incommensurate pleasure in going back to Euclid's *Elements* not to find truth there, but to see how the Greeks conceived of it, much as I come back to Piero della Francesca's paintings to see how the Renaissance conceived of the transcendental realm and pictured the world. I think that we have come to understand that truth is no longer the adequacy of our representations with the objects they represent. Truth is not of the order of adequacy. Truth, as Cornelius Castoriadis argued, "is the constant effort of dismantling the fence in which we find ourselves and to think otherwise, and to think no longer quantitatively, but deeper, better" (Castoriadis 1999, p. 54). Truth would rather be an attitude, what the Greeks called an *ethos*.

Along these lines, let me suggest that maybe we can think of mathematics as a historically constituted social practice, a cultural form of reflection and action, much like music, poetry, or painting, something practiced not in a vacuum but *with* others and *for* others. Mathematics would be hence not something to acquire (as if mathematics were merchandise) but a practice in which we come to insert ourselves, where we step into the public space. It would be what Arendt (1958), following the Greeks, called the *polis*—a place where we come to hear others' voices and perspectives and to speak out. There may be some hope then that in doing so, our students will no longer find themselves in front of an impenetrable alienating discourse, but rather will grow up as subjects of mathematics, as critical cultural subjects.

Let me now move to the question of history and start with the following remark. Something that distinguishes the human species from other species is our historical nature. Indeed, while rats are still doing what they were doing five hundred years ago, individuals are not. We draw on what previous generations have accomplished.

This is why history cannot be merely a tool to make mathematics accessible to our students. History is a necessity. As Russian philosopher Eval Ilyenkov put the matter, history is a necessity because "A concrete understanding of reality cannot be attained without a historical approach to it." (Ilyenkov 1982, p. 212)

Reality, indeed, is not something that you can grasp by mere observation. Neither can it be grasped by the applications of concepts, regardless of how subtle your conceptual tools are. The current configurations of reality are tied, in a kind of continuous organic system, to those historic-conceptual strata that have made reality what it is. Reality is not a thing. It is a process which, without being perceived, discreetly goes back, every moment, to the thoughts and ideas of previous generations. History is embedded in reality and reality in history.

To confine history to a tool for cognitive improvement is certainly a good idea. But would we not be missing the most important point? This is my third provocative question.

History is something that can make us aware of who we are, and how we have come to be the individuals that we are.

Yet, as Brown (2011) reminds us, history is a problematic concept. Indeed, we can ask ourselves: Whose history? Told by whom? History is not something out there. History is not and cannot be an objective, neutral account of events. History is our spatial and temporal situated understanding of something that tells not only the story of some events but also our own story. One of the challenges that we have to face when using history… I am sorry, I do not like the utilitarian expression "using history"… Let me restart my phrase… One of the challenges that we have to face when resorting to the histories (as a plural noun) of mathematics (as a plural noun), is that we have to go beyond the rationalist, regulative view of history that sees it unfolding as naturally as the movement of a pendulum. There is no history, but histories. And histories are political in the sense that we cannot focus on everything and that our histories leave in the margins events, voices, and presences. Bringing in political histories of mathematics in teaching and learning may help us and our students understand that mathematics can only make sense within the context of a history of its own culture; it can help us see how mathematics operates within the centrifugal forces of society, how it accomplishes inclusion and exclusion, how it offers cognitive templates of development, and how it helps to shape the selves into which we evolve in our lives. This is why histories are not narratives of the past. Histories configure our present and make it possible to envision a future that, ironically, is already historical, even if it is unpredictable.

Reflective Summary

The various sections of this chapter point towards challenges and possibilities in resorting to the history of mathematics in mathematics education and mathematics education research. The possibilities are certainly substantial, but so are the challenges. The history of mathematics cannot be simply imported into the classroom, nor can it be used as a transparent mediating term between the poles of mathematics and mathematics education research. To be a meaningful mediator, the history of mathematics needs to appear as a problematic field—one where one can interrogate notions and ideas that we usually leave unthematized, such as mathematics, its development, and its relationship to culture. To be a meaningful mediator, history has to subject itself to an enquiry of its own meaning. Such a task, of course, is extremely difficult. We can only vaguely perceive its contours. At this point, it appears as an *abstract notion* in Hegel's sense: something that has to find determination in the event of its concrete activity. It might be the case that such an endeavor will lead us to a better and deeper appreciation of what mathematics is, and of what teaching and learning mathematics entail. It might be the case that we end up finding new forms of thinking mathematically that have remained buried underneath the thick layers of technicalities and calculations that characterize to a large extent the mathematics of today. Maybe we can still extract from the deepness of today's practice the aesthetic, intersubjective and embodied dimensions of mathematics and mathematical thinking.

References

Arendt, H. (1958). *The human condition*. Chicago: University of Chicago Press.

Barbin, E., Stehlíková, N., & Tzanakis, C. (2008). *History and epistemology in mathematics education*. Prague: Vydavatelský Servis.

Borel, É. (1907). À propos de l'enseignement de la géométrie élémentaire. *Revue Scientifique*, *5*(8), 656–658 (On the teaching of elementary geometry).

Borel, É. (1914). L'adaptation de l'enseignement secondaire aux progrès de la science. *L'Enseignement Mathématique*, *16*, 198–210 (The adaptation of high-school teaching to the progress of science).

Bourlet, C. (1910). La pénétration réciproque des mathématiques pures et des mathématiques appliquées dans l'enseignement secondaire. *L'Enseignement Mathématique*, *12*, 372–387 (The reciprocal penetration of pure and applied mathematics in high school teaching).

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer Academic.

Brown, T. (2011). *Mathematics education and subjectivity: cultures and cultural renewal*. New York: Springer.

Brown, T., & Walshaw, M. (Guest editors) (2012). *Mathematics education and contemporary theory. Educational Studies in Mathematics*, *80*(1–2), 1–285.

Castoriadis, C. (1999). *Dialogue*. La Tour D'Aigues: Éditions de l'Aube.

Châtelet, G. (2000). *Figuring space: philosophy, mathematics, and physics*. Dordrecht: Kluwer Academic.

Corry, L. (2006). Axiomatics, empiricism, and Anschauung in Hilbert's conception of geometry: between arithmetic and general relativity. In J. Ferreirós & J. Gray (Eds.), *The architecture of* *modern mathematics: essays in history and philosophy* (pp. 133–156). Oxford: Oxford University Press.

Dreyfus, T., & Eisenberg, T. (1982). Intuitive functional concepts: a baseline study on intuitions. *Journal for Research in Mathematics Education*, *13*, 360–380.

Dreyfus, T., & Eisenberg, T. (1986). On the aesthetics of mathematical thought. *For the Learning of Mathematics*, *6*(1), 2–10.

Eisenberg, T. (1977). Begle revisited: teacher knowledge and student achievement in algebra. *Journal for Research in Mathematics Education*, *8*, 216–222.

Eisenberg, T., & Dreyfus, T. (1991). On the reluctance to visualize in mathematics. In W. Zimmermann & S. Cunningham (Eds.), *MAA notes: Vol. 19. Visualization in teaching and learning mathematics* (pp. 25–37). Washington: Math. Assoc. of America.

Eisenberg, T., & Fried, M. (2009). Dialogue on mathematics education: two points of view on the state of the art. *ZDM*, *41*, 143–149.

Featherstone, H., & Featherstone, J. (2002). "The word I would use is aesthetic": reading David Hawkins. *For the Learning of Mathematics*, *22*(2), 24–27.

Fried, M. N. (2001). Can mathematics education and history of mathematics coexist? *Science & Education*, *10*, 391–408.

Fried, M. N. (2007). Didactics and history of mathematics: knowledge and self-knowledge. *Educational Studies in Mathematics*, *66*, 203–223.

Freudenthal, H. (1981). Mathematics teacher know something about the history of mathematics? *For the Learning of Mathematics*, *2*(1), 30–33.

Furinghetti, F. (2007). Teacher education through the history of mathematics. *Educational Studies in Mathematics*, *66*, 131–143.

Grabiner, J. V. (1974). Is mathematical truth time-dependent? *The American Mathematical Monthly*, *81*(4), 354–365.

Hulin, N. (2011). *Culture scientifique et humanism*. Paris: L'Harmattan (Scientific culture and humanism).

Ilyenkov, E. V. (1982). *The dialectic of the abstract and the concrete in Marx's Capital*, Moscow: Progress Publishers.

Jahnke, H. N. (2000). The use of original sources in the mathematics classroom. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education: the ICMI study* (pp. 291–328). Dordrecht: Kluwer Academic.

Klein, F. (1896). The arithmetizing of mathematics. *Bulletin of the American Mathematical Society*, *2*(2), 241–249. An address delivered at the public meeting of the Royal Academy of Sciences of Göttingen, November 2, 1895 (Trans. I. Maddison)

Klein, F. (1924). *Elementarmathematik vom höheren Standpunkte aus* (Vol. 1). Berlin: Springer (Elementary mathematics from an advanced point of view).

Lizcano, E. (2009). *Imaginario colectivo y creación matemática*. Madrid: Gedisa (Collective imaginary and mathematical creation).

Loria, G. (1914). *Le scienze esatte nell'antica Grecia*. Milan: Hoepli (The exact sciences in ancient Greece).

Marre, A. (1880). *Bullettino de bibliografia e di storia delle scienze matematiche et fisiche: Vol. 13. Le triparty en la science des nombres de maistre Nicolas Chuquet Parisien* (The triparty of number science of master Nicolas Chuquet, Parisien) (Original work published 1484).

Netz, R. (2009). *Ludic proof: Greek mathematics and the Alexandrian aesthetic*. New York: Cambridge University Press.

Pimm, D., & Sinclair, N. (2009). Audience, style and criticism. *For the learning of mathematics*, *29*(2), 23–27.

Poincaré, H. (1899). La logique et l'intuition dans la science mathématique et dans l'enseignement. *L'Enseignement Mathématique*, *1*, 157–162 (Logic and intuition in the mathematics science and teaching).

Radford, L. (2004). From truth to efficiency: comments on some aspects of the development of mathematics education. *Canadian Journal of Science, Mathematics and Technology Educa-*

tion/Revue Canadienne de l'enseignement des Sciences, des Mathématiques et des Technologies, *4*(4), 551–556.
Radford, L. (2008). Culture and cognition: towards an anthropology of mathematical thinking. In L. English (Ed.), *Handbook of international research in mathematics education* (2nd ed., pp. 439–464). New York: Routledge/Taylor & Francis.
Rotman, B. (2008). *Becoming beside ourselves: the alphabet, ghosts, and distributed human beings*. Durham: Duke University Press.
Sinclair, N. (2001). The aesthetic is relevant. *For the learning of mathematics*, *21*(1), 25–33.
Solomon, Y., & O'Neill, J. (1998). Mathematics and narrative. *Language and Education*, *12*(3), 210–221.
Spiesser, M. (2006). L'algèbre de Nicolas Chuquet dans le contexte français de l'arithmétique commercial. *Revue d'histoire des Mathématiques*, *12*, 7–33 (Nicolas Chuquet's algebra in the French context of commercial arithmetic).
Tall, D. (2003). Using technology to support an embodied approach to learning concepts in mathematics. In L. M. Carvalho & L. C. Guimarães (Eds.), *Historia e tecnologia no ensino da matemática* (Vol. 1, pp. 1–28).
Unguru, S. (1975). On the need to rewrite the history of Greek mathematics. *Archive for History of Exact Sciences*, *15*, 67–114.
van der Waerden B.L. (1976). Defence of a 'shocking' point of view. *Archive for History of Exact Science*, *15*, 199–210.