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PERCEPTUAL SEMIOSIS AND THE MICROGENESIS OF ALGEBRAIC GENERALIZATIONS[1]

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Abstract: *This paper deals with the problem of algebraic generalizations of elementary geometric-numeric patterns. It focuses on understanding the role played by the various semiotic systems mobilized by students in the progressive process of perceptual apprehension of a pattern and its generalization. The microgenetic analysis of the mathematical activity of two small groups of students in a Grade 9 class shows how making recourse to semiotic resources, such as gestures, language, and rhythm, allows the students to objectify different aspects of their spatial-temporal mathematical experience. The analysis also shows some connections between the syntax of the students' algebraic formulas and the semiotic means of objectification through which the formulas were forged, thereby shedding some light on the meaning of students' algebraic expressions.*

Keywords: generalization, gestures, meaning, objectification, rhythm, semiotics, semiotic-cultural approach, signs, syntax.

INTRODUCTION AND THEORETICAL FRAMEWORK

Resorting to a small number of characters to form an expression, algebraic symbolism allows us to convey an idea that, usually, in natural language, may take one or several lines. Algebraic symbolism does not possess the rich arsenal of resources such as adverbs, adjectives and noun complements that play a crucial role in written and oral languages. Instead, it offers to its users a precision and succinctness governed by a few syntactic rules. However, the ability to grasp how this precision and succinctness works often becomes difficult for students of different ages, as is reflected by the large amount of research devoted to the understanding of students' errors (see e.g. Matz, 1980; Kaput and J. Sims-Knight, 1983). Regardless of their theoretical orientation, the research results agree on this: algebraic syntax is not transparent.

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In our previous research, we have focused on the investigation of the meaning with which students endow elementary algebraic expressions. Our research has been encompassed by a semiotic-cultural perspective that rests on the idea that learning is accomplished through the use of diverse semiotic systems. Indeed, even accurate discourse is unable to lead students directly to the object of learning, for learning is entailed by meaning and interpretation. Thus, to learn to generalize geometric-numeric patterns amounts to learning to see and to interpret a finite number (usually very few) of concrete objects or signs in a different way. To learn to generalize means to "notice" (Mason, 1996) something that goes beyond what is actually seen. Ontogenetically speaking, this act of noticing unfolds in a gradual process in the course of which the object to be seen emerges progressively. This process of noticing we have termed a process of *objectification*. To make something apparent (which is the etymological sense of objectification) learners and teachers make recourse to signs and artefacts of different sorts (mathematical symbols, graphs, words, gestures, calculators, and so on). These artefacts and signs used to objectify knowledge we call *semiotic means of objectification* (Radford, 2003).

One of the basic principles of our methodological approach to the investigation of the students' algebraic generalizations can be stated as follows. Our comprehension of the meaning with which the students endow their algebraic expressions can be deepened by investigating the semiotic means of objectification to which the students have recourse in their attempt to accomplish their generalizations.

This methodological principle is interwoven with the theoretical tenet of our research approach mentioned above, namely, that learning is essentially a social process of objectification mediated by a multi-systemic semiotic activity.

In previous works we have discussed the prominent role of gestures and language in students' processes of knowledge objectification. We have provided evidence of the key role of deictic activity, both at the level of gestures (like in pointing) and at the level of language (e.g. when students use terms such as *this* and *that*)[2]. In more precise terms, in our study of students' semiotic mechanisms through which the mathematical structure of a pattern is revealed, we have found a rich process of objectification in which the mathematical structure of the pattern is ostensibly asserted by gestures and linguistic key terms (Radford 2002, 2003).

Often, the students' objectification of the conceptual categories required in the generalization of patterns takes the form of a process of *perceptual semiosis*, *i.e.* a process relying on a use of signs dialectically entangled with the way that concrete

[2] By deictic activity we mean the activity embedded in social communicative processes where actions (e.g. pointing gestures), linguistic units (e.g. 'top', 'bottom'), etc. allow one to refer to the objects in the universe of discourse. It is the contextual circumstances which determine their referents. As such, deictic terms depend heavily on the context (see Nyckees 1998, p. 242 ff.) and have a particular function in dialogical processes.

objects become perceived by the students. In this paper we want to deepen our analyses in order to better understand the students' processes of perceptual semiosis. We are interested in understanding the dialectical relationships between the various semiotic systems mobilized by the students in making sense of generality as expressed through algebraic symbolism. We will focus on the work of two small groups of 3 students each, during a regular Grade 9 mathematics lesson. In the next section, we describe the task and some elements of the mathematical lesson and of our methodology.

METHODOLOGY

The data reported here comes from a 5-year longitudinal study, collected during classroom activities. The activities are part of the regular school teaching lessons, as framed by the Ontario provincial Curriculum of Mathematics. In these activities, the students spend a substantial period working together in small groups of 3 or 4. At some points, the teacher (who interacts continuously with the different groups during the small group-work phase) conducts a general discussion allowing the students to expose, confront and discuss their different solutions. In addition to collecting written material, tests and activity sheets, we have three or four video-cameras each filming one group of students. Subsequently, transcriptions of the video-tapes are produced. Video-recorded material and transcriptions allow us to identify salient short passages that are then analyzed using techniques of qualitative research in terms of the students' use of semiotic resources (details in Radford, 2000).

The mathematical problem on which we will focus here was the first one of a three-problem math lesson. This problem dealt with the study of an elementary geometric sequence (see Figure 1). The students were required to continue the sequence up to Figure 5 and then to find out the number of circles on figures number 10 and number 100. Subsequently they were asked to write a message indicating how to find out the number of circles in any figure (*figure quelconque*, in the original French), and then to write an algebraic formula for the number of circles in figure number *n*.

Figure 1

In the next section, we discuss two examples of perceptual semiosis and the role played by the latter in the students' elaboration of their algebraic formulas. While one of the processes of perceptual semiosis led to the formula "(n+1) + (n+2)", the second process led to the formula "n×2+3". As we shall see, the study of the microgenesis of students' generalizations provides us with rich information about the meaning of students' algebraic symbolism. It will become apparent that the students' apprehension of the pattern and the building of generality are underpinned by a complex articulation of written signs, words and gestures.

PROCESSES OF PERCEPTUAL SEMIOSIS

First example:

The first group is formed by three students, Doug (left), Alice (center), and Mireille (right)[3]. After drawing figures 4 and 5, Doug says:

"So we just add another thing, like that" *(when he utters the last word he makes a sequence of gestures*).

Figure 2. Excerpt of Doug's sequence of rhythmic gestures.

Since the word 'thing' does not have a clear referent, Doug immediately adds the expression "like that". Interestingly, the deictic 'that' does not refer to something concrete on the sheet where the figures have been drawn, but to something else, something that is ostensibly shown by a rhythmic sequence of six gestures iconically suggesting inclined lines (see Figure 2). Doug's ostensible mechanism serves two purposes: (1) to orient the process of perceptual semiosis in a certain direction (here, emphasizing the last two circles on each row), and (2) to convey a sense of generality through the rhythm of the gestures. In fact, the six diagonals virtually drawn by Doug with his rhythmic gestures not only refer to the last two circles diagonally disposed at the end of each figure but also express the idea of something that spatial-temporally continues further and further.

When solving the problem of finding the number of circles in figure number 10, the regularity of the pattern is reformulated: what was previously perceived as a unique object (the couple of two circles) is now atomised (two separated circles). While *drawing* figures 5 and 6 did not require knowing the number of circles in each of these figures, this knowledge became essential for solving the next question that the students solved by *computation*.

This atomisation is then soon refined by Alice, who suggests another way of expressing the regularity, based on another perception of the figures. Now the figures are seen as divided into two rows:

[3] Names have been changed for deontological reasons.

1. Alice: No, you just have to always add one on the top and one on the bottom (*inclining her head towards the right when she says "bottom"*).
2. Doug: Umm. OK. So it's ... [...] How many... How many circles will figure number ten have?
3. Alice: OK. It would be (*pointing with her finger the rows of figure 2*) eleven on the top and then...and then... twelve on the bottom.

Alice mobilises two semiotic resources to objectify her perception of the figures and, by the same token, to refine her understanding of the regularity of the pattern. When talking in general terms ("you (...) always", line 1), the distinction between the two rows is made by the inclination of her head, meant to clearly distinguish the circle added on the top from the one added on the bottom. Later on, when tackling the problem for figure number 10 (line 3), the distinction between rows is made by pointing at the top row of figure 2. This figure indeed provides the students with a *metaphoric* way of talking meaningfully about figure 10; it is a concrete support for them to *imagine* figure 10.

The shift towards Alice's perception of the figure (*i.e.* in two rows) is not problematic for Doug, who soon agrees with her point of view ("Umm. OK", line 2). This point of view allows the students to easily find the number of circles in figure number 10, as well as the ones in figure number 100.

When asked to write a message in French, describing how to explain to another student what she/he should do in order to find the number of the circles in any figure, Doug says:

4. Doug: Each... For each figure… You take the number of the figure...of the...of the... The number of the figure (*balancing nervously back and forth on his chair*) […] (*then, without balancing anymore he says*) let's say that the figure's number is three. You would say one plus three for the top row (*moving his pencil in the air from left to right*) and two plus three... […] No, plus two for the bottom row (*pointing with his finger at one of the figures*) and plus one (*pointing directly to one of the figures*) for the top row. On …of the number... the figure (*stressing the words "on" and "of" by pointing his finger towards the table)*.

Doug does not seem to be comfortable dealing with the problem of "any figure". He is not comfortable in this layer of generality and expresses himself hesitantly, moving on his chair nervously. After the early unsuccessful attempts, Doug abandons this path to generality and uses figure 3 as a crutch. The concreteness of figure 3 allows him to express the general intended computations. As soon as he finishes explaining the computations based on figure 3, the reference to a particular figure fades away (Doug even says "no", as if he were making a mistake). In actual fact, he is not talking about figure 3 specifically. In Kantian terms, the counting process undertaken on figure 3 serves as a way to objectify the *schema of counting*. Doug's effort shows us at least this:

the presence of the general is made apparent by its <u>*absence*</u> *at the discursive level*: “plus two” and “plus three” do not have an explicit linguistic referent and the gesture in the air signifies that the referent is not located on the drawn figures either. By omitting to name the referent, the referent becomes general.

Although Doug’ utterance evokes a certain “struggle” (Doug has some troubles, at the end, to make his sentence coherent by trying to include the reference to the figure again), the written message in plain French is quite clear:

> “# de fig +1 pour la rangée du haut et # de la fig +2 pour le bas.
> Additionne les deux pour le total”[4].

The written message tells us more than the sole description of the procedure that one has to perform in order to find out the number of circles in any figure. It also states the geometrical meaning of the operations, intimately related to the students’ perception of the figure into two rows. The need to refer to the geometrical meaning can also be found elsewhere in their answers, more specifically when the students answer the second and third questions: “23 circles (11 on the top, 12 on the bottom)” and “203 circles (101 on the top, 102 on the bottom),” respectively.

The algebraic formula that they provide at the end of the problem (*i.e.* “(n+1)+(n+2)”) still follows the course of this geometrical explanation, where the brackets delimitate the computations made on the two rows of the figure. Brackets are organizers of the way in which the formula tells us the story of the students’ mathematical experience.

Second example:

The group is formed by three students: Jay, Mimi (sitting side by side) and Rita (sitting in front of them). The students begin counting the number of circles in the figures, realizing that the number of circles in the figures increase by two each time. Then, their attention focuses on the geometrical structure of the figures:

1. Mimi: *(Talking about figure 4)* So, yeah, you have five on top *(she points to the sheet, sketching a horizontal line with her hand)* and six on the... *(she points to the sheet, making anther horizontal gesture, lower that the previous one)*.
2. Jay: Why you’re putting... oh yeah yeah yeah, there will be eleven, I think.
3. Rita: Yep.
4. Mimi: But you must go six on the bottom … and five on the top.

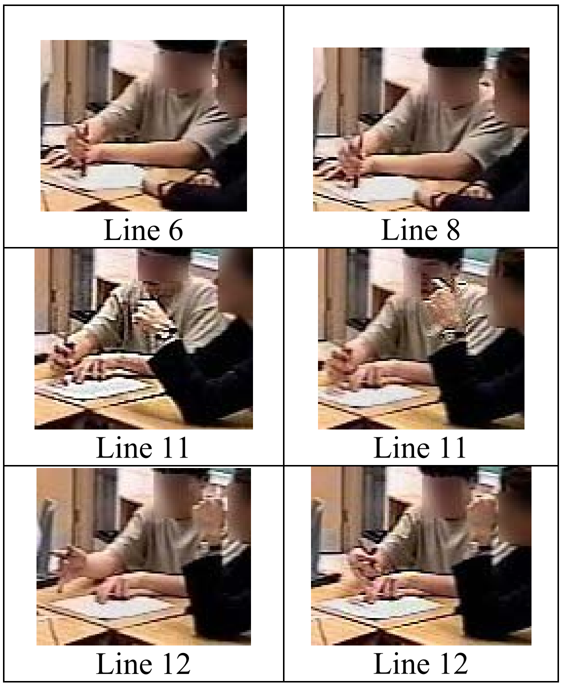
The spatial deictics “top” and “bottom” (lines 1 and 4) used by Mimi offer her group-mates a particular way to apprehend the figures in the ongoing process of perceptual semiosis. Jay’s utterance (line 2) reminds us that, despite what is often thought, perceiving is not a simple and direct process. In line 4, Mimi insists on the geometric structure of the terms of the sequence. Her intervention amounts to shifting from blunt

[4] Transl: “# of fig +1 for the top row, and # of the fig +2 for the bottom. Add the two to get the total”.

counting to a scheme of counting. To notice this scheme is the first step towards the general.

Mimi's spatial deictics are accompanied by two corresponding gestures. These gestures accomplish a twofold role: they depict the spatial position of the rows in an iconic way and also clarify the reference of the uttered words.

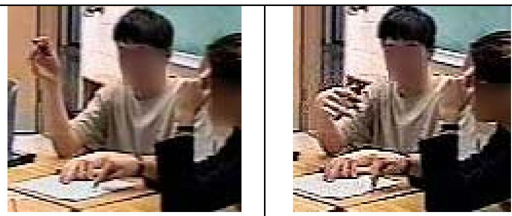
The students' work was interrupted by an announcement made to the class about a forthcoming social activity. While Mimi and Rita paid attention to the announcement, Jay kept on working, writing 23 and 203 as the answers for the question concerning the number of circles in figures 10 and 100. At this point, the announcement was finished and the girls returned to the task and asked Jay for an explanation:



Line 6 | Line 8 | Line 11 | Line 11 | Line 12 | Line 12

5. Mimi: (*Talking to Jay*) I just want to know how you figured it out.
6. Jay: Ok. Figure 4 has five on top, right? *(he points to the top row of figure 4)*.
7. Mimi: Yeah…

8. Jay : *(Continuing his sentence)* and it has 6 on the bottom *(he points to the bottom row)* […].
9. Mimi: *(She points to the circles while she counts)* 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. (*Pause*) […] Oh yeah. Figure 10 would have …
10. Jay: 10 there would be like…
11. Mimi: There would be eleven *(she makes a quick gesture that points to the air)* and there would be ten *(same quick gesture but higher up)* right?
12. Jay: Eleven (*similar gesture but with the whole hand on the paper*) and twelve *(same gesture but lower)*.
13. Mimi: Eleven and twelve. So it would make twenty-three, yeah.
14. Jay: [Figure] 100 would have one-hundred and one and one-hundred and two *(same gestures as in line 12)*.
15. Mimi: OK. Cool. Got it now. I just wanted to know how you got that.



Line 14 Line 14

Figure 3. Some gestures occurring in the lines of the dialogue.

Developing Mimi's initial idea, elaborated in lines 1 and 4, Jay here attains a structural apprehension of the figure through which he solves the problem for figures 10 and 100. Let us notice that, to explain his strategy (lines 6, 8), he refers first to figure 4. In line 7 agreement is obtained. Moreover, in his explanation, he uses the same discourse genre as Mimi's: a discourse genre that interweaves spatial deictics (top, bottom) and iconic and deictic gestures. In particular, by pointing gestures he touches the two horizontal rows in which figure 4 can be divided. Mimi then turns to figure 10 (end of line 9) and accompanies her utterance with gestures that keep certain specific aspects of those of Jay: the fact of having one gesture for each row. But whereas Jay's gestures point materially to the rows of figure 4, Mimi's are made in the air (line 11): indeed, figure 10 is not in the perceptual field of the students, so new mechanisms of semiotic objectification have to be displayed. This, we suggest, is the role of gestures here. Of course, Mimi could have simply reached the answer using words. The fact that she did not, and that she used gestures is right to the point that we want to make here: gestures do not merely carry out intentions or information. They are key elements of the process of knowledge objectification. This point becomes even clearer when the students address the question of figure 100. The gestures are again made in the air, and this time at a higher elevation from the desk.

In their path towards generality, students need to mobilize both language and gestures in a coordinated and efficient way. This coordination takes place in particular segments of the students' mathematical activity where knowledge is objectified. These segments of mathematical activity characterized by the crucial coordination of various semiotic systems constitute what we have previously termed *semiotic nodes* (Radford et al. 2003).

In this particular semiotic node, which goes from line 5 to 15, we see how the students merge the geometrical and numerical components of the problem: the former is taken in charge by gestures and the latter by words.

We shall now discuss the part of the mathematical lesson where the students had to face the problem of writing a message to explain how to find out the number of circles for any figure (*figure quelconque*).

16. Mimi: Add. Add three to the number of the figure! (*pointing to the results 23 and 203 on the paper*).
17. Jay: No! […].
18. Mimi: I mean like … I mean like … You know what I mean, like, for figure 1 […] (*pointing to figure 1*) it would be like one, one, plus three; this *(pointing to figure 2)* would be two, two, plus three; this *(pointing to figure 3)* would be three, three, plus three.

As suggested by her gestures (line 16), Mimi seems to have observed that the number of circles in figures 10 and 100 ended with the digit 3 and considered it as a key to look for a general method, something which led her to a new apprehension of the figure. Jay does not understand (line 17). Mimi then explains the idea in more detail (line 18). Here, the gesture with which she pointed to each figure was made up of three indexical gestures. In the case of figure 1, she pointed successively to the top left circle, then the bottom left circle and finally she sketched a small triangle surrounding the three left circles on the right (see Figure 4).

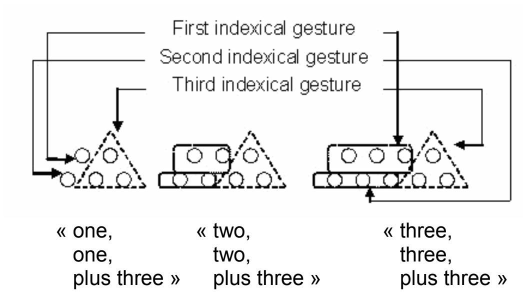


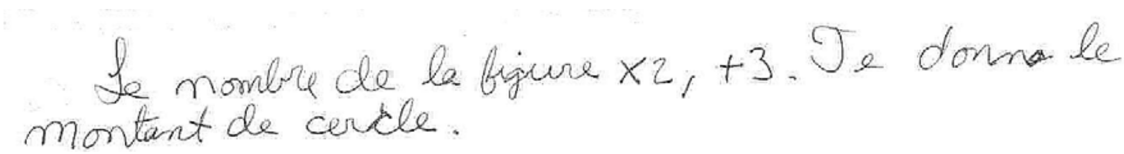
Figure 4. On the left, Mimi making the indexical gestures on figure 1 on the sheet. On the right, the new apprehension of the figures as a result of the process of perceptual semiosis.

The process of perceptual semiosis leading to the new apprehension of the structure of the figures included not only gestures and words, but also rhythm. In fact, the expression "one, one, plus three" is uttered with the same cadence as the expressions "two, two, plus three" and "three, three, plus three". We can detect, in this sentence, the embedding of two types of rhythm. The first one helps to make apparent a kind of regularity within each figure, and in conjunction with gestures and the meaning of words, organizes the way of counting. The second rhythm corresponds to the *pause* made *between* figures: “one, one, plus three” [*pause*] “two, two, plus three” [*pause*] “three, three, plus three”. The concatenation of these two rhythms conveys the idea of generality. It also opens new avenues to keep exploring the general. Thus, in the course of the classroom activity, it became apparent that the first two elements in the counting process were related to the number of the figure.

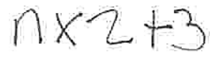
In fact, keeping the numerical example of figure 10, the students soon after manage to express the regularity in natural language:

19. Mimi: You double the number of the figure.
20. Jay: 10+10
21. Mimi: So it will be 20 dots +3. You double the number of the figure and you add three, right. So figure 25 will be 50...3. Right? That’s what it is.

The message was finally refined as follows:



The symbolic formula was:



SYNTHESIS AND CONCLUSION

Our microgenetic analysis of two small groups of students dealing with the generalization of patterns suggests the central role played by spatial deictics, gestures, and rhythm in perceptual semiosis, particularly in the students’ progressive processes of perceptual apprehension of a pattern and its generalization. The analysis also suggests some connections between the syntax of the students’ algebraic formulas and the students’ semiotic means of objectification. For instance, the spatial deictics ‘top’ and ‘bottom’ impressed their mark in the syntax of the formula “(n+1)+(n+2)”. However, the connection may be even yet more subtle. Rhythm, for example, impressed its mark in the message produced by the second group of students, where it appeared under the form of a comma (see above). In the final symbolic formula “n×2 + 3” the comma has disappeared. Rhythm is nevertheless *embedded* in the symbolic expression: it constitutes one of the signifying elements of the students’ formula. In general, deictics, gestures, rhythm, and other semiotic means of objectification do not operate separately from each other. They belong to different semiotic systems whose coordination seems to be crucial in the students’ mathematical experience. This complex coordination of semiotic systems still remains largely unknown in the psychology of mathematics education. This paper does not solve this research problem in its generality. It shows a few elements of it and suggests a research path to be explored.

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