Plenary Lecture

THE CULTURAL-EPISTEMOLOGICAL CONDITIONS OF THE EMERGENCE OF ALGEBRAIC SYMBOLISM[1]

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> I have often taken up a book and have talked to it and then put my ears to it, when alone, in hopes it would answer me: and I have been very much concerned when I found it remained silent.
> *The interesting narrative of the life of O. Equiana* (Cited by M. Harbsmeier, 1988, p. 254)

ABSTRACT

The main thesis of this paper is that algebraic symbolism emerged in the Renaissance as part of a new type of thinking – a new type of thinking shaped by the socioeconomic activities that arose progressively in the late Middle-Ages. In its shortest formulation, algebraic symbolism emerged as a semiotic way of knowledge representation inspired by a world substantially transformed by the use of artefacts and machines. Algebraic symbolism, I argue, is a metaphoric machine itself encompassed by a new general abstract form of representation and by the Renaissance technological concept of efficiency. To answer the question of the conditions which made possible the emergence of algebraic symbolism, I enquire about the cultural modes of representation of knowledge and human experience and look for the historical changes which took place in cognitive and social forms of signification.

1 Introduction

The way in which I wish to study the problem of the emergence of algebraic symbolism can easily lead to misunderstandings. Perhaps the most tempting misunderstanding would be to think of this paper as a historical investigation of the external factors that made possible the rise of symbolic thinking in the Renaissance. "External factors" have usually been seen as economic and societal factors that somewhat influence the development of mathematics. They are opposed to "internal factors", which are seen as the true factors accounting for the development of mathematical ideas. The distinction between the internal and external dimensions of the conceptual development of mathematics rests on a clear cut distinction between the sociocultural on one side, and the "really" mathematical on the other. Within this context, the former is seen, as Lakatos suggested, as a mere *complement* to the latter. Viewed from this perspective, it may appear that the route I am taking to investigate the emergence of algebraic symbolism belongs to the sociology of knowledge. However, to cast my intentions in such a dichotomy is misleading.

On the one hand, current research on human cognition is emphasizing the tremendous role played by the context in the concepts that we form about the world. As Otte (1994, p. 309) summarized the idea, "The development of knowledge does not take place within the framework of natural evolution but within the frameworks of sociocultural developments." Thus, if we want

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to understand the mathematical ideas of a certain historical period, we need to understand their encompassing sociocultural developments in the amplest sense.

On the other hand, in the past few years, more and more arguments have been produced to the effect that mathematics bears the imprint of its culture, so that, under closer examination, what seemed to be "external" is not. As Crombie (1995, p. 232) noted, the cultural conception of mathematics determines the organization of scientific inquiry, the kind of arguments that will be socially accepted, the kind of evidence and the type of explanations that will be considered valid.

The awareness that there may be a relationship between mathematical thinking and its own cultural context has moved current historical and epistemological discussions away from naturalist and rationalist accounts of mathematical thinking. However, the awareness of the relationship between culture and thinking is not enough. As a matter of fact, historical and epistemological accounts of mathematical conceptual developments have thus far not been very successful in specifying how mathematical thinking relates to culture. I want to go further and suggest that if we do not specify the link between culture and mathematical conceptualizations, we risk using culture as a generic term that attempts to explain something, while in reality it does not explain anything.

In the first part of this paper, I will outline the theoretical framework to which I will resort in order to attempt to answer the question of the conditions of the emergence of algebraic symbolism. In the second part, I will deal with the place of algebra in its historical setting, focusing mainly on changes in the cultural forms of signification and knowledge representation.

2 The link between culture and knowledge

The Semiotic Anthropological Perspective that I have been advocating[2] draws from the socio-historical school of thought developed by Vygotsky, by Leont'ev's *Theory of Activity* and from Wartofsky's and Ilyenkov's epistemologies[3]. In this perspective, mathematics is considered to be a human production. This claim is consonant with claims made by Oswald Spengler (1917/1948) almost one century ago and revitalized by contemporary scholars such as Barbin (1996), D'Ambrosio (1996), Restivo (1992, 1993), Høyrup (1996, 2002).

There are three key interrelated elements underpinning the Semiotic Anthropological Perspective:

- The concept of activity as a unit of analysis.
- A reconceptualization of knowledge.
- A cultural definition of thinking.

The concept of Activity:

Activity, as a unit of analysis for the understanding of conceptual developments, refers not only to *what* mathematicians were doing at a certain historical moment and *how* they were doing it. It also refers to the ineluctably sociocultural embeddedness of the ways in which mathematics is carried out. Activity, as understood here, emphasizes the culturally grounded "rational" inquiry that constitutes the particularities of mathematical thinking in a certain historical period and setting.

The concept of activity does not tell us, however, in which sense we have to understand the link between culture and knowledge. What we have asserted about activity is good enough for

[2] Radford (1997, 1998, 1999, 2003a).

[3] See Vygotsky (1962, 1978, 1981), Leont'ev (1978), Wartofsky (1979), Ilyenkov (1977).

conceiving of mathematics as a human endeavour, but it is certainly insufficient to bring us beyond the internal/external dichotomy of classical historiography. In other words, the idea of activity expounded thus far provides room for seeing "connections" between mathematical knowledge and its cultural settings, but in no way tells us the *nature* of such "connections". Without further development, the "connections" cannot be *explained* but only empirically *shown*[4].

A reconceptualization of knowledge.

What then exactly is the relationship between culture and knowledge? In opposition to Platonist or Realist epistemologies, knowledge is not considered here as the discovery of something already *there*, preceding human activity. Knowledge is not about pre-existing and unchanging objects. Knowledge relates to culture in the precise sense that the objects of knowledge (geometric figures, numbers, equations, etc.) are the *product* of human thinking. Knowledge is *generated* through sociocultural activities. The way in which knowledge is generated and the very nature of the content of knowledge are related to the sensuous forms of these activities and the historical embodied beliefs and intelligence kept in them. The Pythagorean knowledge about numbers, for instance, was generated in the course of the social-intellectual activities of the brotherhood, mediated by the sensuous use of stones and other mathematical signs to represent knowledge and the historical, cultural, ontological belief that there was a link between the nature of numbers and the universe (Radford, 1995, 2003a).

A cultural definition of thinking.

Following Wartofsky (1979), I conceive of thinking as a cognitive *praxis*. More precisely, thinking, I want to suggest, is a *cognitive reflection of the world in the form of the individual's culturally framed activities*.

As we can see from the previous remarks, activity is not merely the space where people get together to do their thinking. The essential point is that the cultural, economic and conceptual formations underpinning knowledge-generating activities impress their marks on the theoretical concepts produced in the course of these activities. Theoretical concepts are reflections that reflect the world in accordance to the social processes of meaning production and the conceptual cultural categories available to individuals.

What I am suggesting in this paper is that algebraic symbolism is a semiotic manner of reflecting about the world, a manner that became thinkable in the context of a world in which machines and new forms of labor transformed human experience, introducing a systemic dimension that acquired the form of a metaphor of efficiency, not only in the mathematical and technical domains, but also in aesthetics and other spheres of life.

In the next section, I will briefly discuss some cultural-conceptual elements of abacist algebraic activity. In the subsequent sections, I will focus on the technological and societal elements which underlined the changes in Renaissance modes of knowledge representation.

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[4] This is the case with Eves' book *An Introduction to the History of Mathematics*. In contrast to the previous editions of the book (see e.g. Eves, 1964), in the 6th edition (see Eves 1990), a section was added in which the cultural setting was expounded before each chapter. Connections are shown rather than explained. That Netz (1999) placed the cultural aspects of Greek mathematics in the *last* part of his otherwise enlightening book, after all the mathematical aspects were explained (as if the cultural aspects were independent of or at least not really a part of mathematical thinking), is representative, I believe, of the difficulty in tackling the theoretical problem of the connection between culture and mathematical knowledge.

3 Abacist algebraic activity

In his work *Trattato d'abaco,* Piero della Francesca deals with the following problem:

> A gentleman hires a servant on salary; he must pay him at 25 ducati and one horse per year. After 2 months the worker says that he does not want to remain with him anymore and wants to be paid for the time he did serve. The gentleman gives him the horse and says: give me 4 ducati and you shall be paid. I ask, what was the horse worth? (Arrighi (ed), 1970, p. 107)

This is a typical problem from the great number of problems that can be found in the rich quantity of Italian mathematical manuscripts that abacus teachers wrote from the 13th century onwards. This problem conveys a sense of the kinds of reflections in which the Italian algebraists were immersed as a result of the new societal needs brought forward by changes in the forms of economic production. While in feudal times the main form of property was land and the serfs working on it, and while agricultural activities, raising cattle and hunting, were conducted in order to meet the essential requirements of life, during the emergence of capitalism, the fundamental form of property became work and trade (see Figure 1)

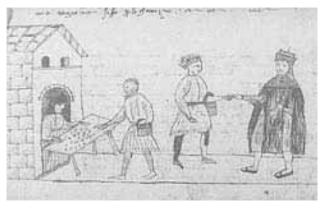

Figure 1. To the left, a man is planting peas or beans, following the harrow (from *Life in a Medieval Village,* F. & G. Gies, 1990, p. 61). To the right, merchants selling and trading products (from Paolo dell'Abbaco's 14th Century *Trattato d'Aritmetica,* Arrighi (ed.), 1964).

Changes in the form of human labor gave rise to new conceptual demands, requiring new cognitive abilities to cope with the various economic practices and new aspects of life. Let us see how della Francesca solved this problem. Note that, to represent the unknown quantities, in some parts of the text, della Francesca uses the term "thing" (*cosa*); in other parts he uses a little dash placed on top of certain numbers. Historically speaking, della Francesca's symbolism is in fact one of the first known 15th Century algebraic symbolic systems.

> Do this. You know that he has to give him 25 ducati per year, for 2 months it comes to 4 I/6; and the horse put that it's worth $\bar{I}$ thing, for 2 months it is worth 2/12 of the thing that is I/6 (*sic*). You know that you have to have in 2 months 4 ducati and I/6 and I/6 of the thing. And the gentleman wants 4 ducati that added to 4 I/6 makes 8 I/6. Now, you have 1/6 of the thing, [and] until $\bar{I}$ there are 5/6 of the thing; therefore 5/6 of the thing is equal to 8 I/6 number. Reduce to one nature [i.e. to a whole number], you will have 5 things equal to 49; divide by the things it comes out to 9 4/5: the thing is worth so much and we put that the horse is worth $\bar{I}$, therefore it is worth 9 ducati 4/5 of a ducato. (Arrighi (ed), 1970, p. 107).

I will come back to the question of symbolism in the next section. For the time being, I want to comment on two of the key concepts involved in the problem: *time* and *value*.

Time: Time appears as a mathematical parameter against which labor is measured. Although time is a dimension of human experience with which cultures have coped in different ways, here we see that the quantification of the labor value (as money loaned at interest in other problems, etc.) requires a strict quantification of time. It requires conceiving of time in new quantifying terms (a detailed discussion about the quantification of time can be found in Crosby, 1997).

Value: Equally important is the fact that summing labor with animals, as Piero della Francesca does here, requires a formidable abstraction. It requires seeing labor (an already abstract concept) and animals (which are tangible things) as homogeneous, at least in *some respect*[5].

As I argued in a previous article (Radford, 2003b), what makes the sum of a horse and labor possible is one of the greatest mathematical conceptual categories of the Renaissance –the category of *value*, a category that neither the abacists nor the court-related mathematicians (see Biagioli, 1989) theorized in an explicit way. Value is the top element in a concatenation of cultural conceptual abstractions. The first one is "usage value". The usage value U(a) of a thing a is related to its "utility" in its social and historical context. The second one is the "exchangeable value"; it puts in relation two usage values and as such it is an *equality* between two different things, something like U(a) = U(b). The third one is of the "value" V(a) of a thing a measured, as in the problem, in terms of money. Value is what allowed individuals in the Renaissance to exchange wax, not just for wool, but for other products as well, and what allowed them to imagine and perform additions between such disparate objects as labor and horses[6].

Value is one of the crystallizations of the economic and conceptual formations of Renaissance culture. As with all cultural categories, value runs throughout the various activities of the time. It lends a certain *form* to activities, thereby affecting, in a definite way, the very nature of mathematical thinking, for thinking –as we mentioned before– is a reflection of the world embedded in, and shaped by, the historically constituted conceptual categories that culture makes available to its individuals.

Horses and labor can be *seen* in the 15th Century as homogeneous because both have become part of a world that appears to its individuals in terms of commodities. They are thought of as having a similar abstract *form* whose common denominator is now money. It makes sense, then, to pose problems about trading and buying in the way it was done in the Renaissance, for money had already become a metaphor, a metaphor in the sense that it stored products, skill and labor and also translated skill, products and labor into each other (see McLuhan, 1969, p. 13).

What does all this have to do with algebra? We just saw that value was the central element allowing individuals in the Renaissance to establish a new kind of abstract relationship between different things. In terms of **representations**, value made it possible to see that one thing could

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[5] To better appreciate the abstraction underpinning the homogeneous character with which two different commodities such as labor and animals are considered in the previous problem, it is worthwhile to recall the case of the Maoris of New Zealand, for whom not all things can be included in economic activity. As Heilbroner reminds us, "you cannot ask how much food a bonito hook is worth, for such a trade is never made and the question would be regarded as ridiculous." (Heilbroner, 1953/1999, p. 27).

[6] Of course, money as the concrete expression (i.e. the sign) of value was used in ancient civilizations such as Mesopotamia, Egypt and Greece (Rivoire, 1985; Sédillot, 1989). However, during the Renaissance, money is no longer simply a convention as it was for Aristotle and Athenian society (see Hadden, 1994; Radford, 2003b). During the period of emergent capitalism, money was conceived of as belonging to the class of things coming from nature and from the work of individuals. Thereby, it was possible to conceive of things as being, in a sense, homogenous. (For additional details about the cognitive impact of commodity exchange activities see the classical work of Sohn-Rethel, 1978. Sohn-Rethel rightly pointed out the kind of abstraction that emerges from commodity production but, in a move coherent with historical materialism, went too far to reduce cognition to the economic sphere. Indeed, this move leads one to a too reductive picture of human cognition. See Radford, in press).

take the place of another, or, in other terms, that one thing (a money coin, e.g.) could be used to **represent** something else. And this is the key concept of algebraic representation.

However, although the conceptual category of value was instrumental in creating new forms of signification and of representation, the concept of value cannot fully account for the emergence of algebraic symbolism. To be sure, value was instrumental in creating different new forms of signification which were distinct from medieval ones (which were governed by iconicity or figural resemblance, or those mentioned by Foucault (1996), like *convenientia* and *aemulatio*, or *analogie* and *sympathie*). Without a doubt, value has shown that representation is arbitrary in the sense that the value of a thing does not reside in the thing itself but in a series of contextual usage values, and we know that the arbitrariness of the signifier is one of the key ideas of algebraic representation. But I will argue later that, along with value, there was another cultural category that played a fundamental role, too. I will come back to this point shortly. Let us now deal with what I want to term *oral algebra*.

4 Oral algebra

As Franci and Rigatelli (1982, 1985) have clearly shown, algebra was a subject taught in the abacus schools. Algebra was in fact part of the advanced curriculum of merchants' education. As in the case of the other disciplines, the teaching and learning of algebra was in all likelihood done for the most part orally. The abacists' manuscripts, which were mostly intended as teachers' notes, indeed exhibit the formulaic texture of oral teaching. They go from problem to problem, indicating, in reasonable detail, the steps to be followed and the calculations to be performed.

Let us come back once more to della Francesca's problem. The text says:

> Do this. You know that he has to give him 25 ducati per year, for 2 months it comes to 4 I/6; and the horse put that it's worth Ī thing, for 2 months it is worth 2/12 of the thing that is I/6 (sic).

From the text, we can easily imagine the teacher talking to one student. When the teacher says "Do this" he uses an imperative mode to call the student's attention to the order of the calculations that will follow. Then, he says: "You know that …". The colloquial style of face to face interaction is indeed a common denominator of abacists' manuscripts[7]. In all likelihood, oral explanations were accompanied by the writing of calculations. This is suggested by the use of the recurrent imperative accompanying the algebraic symbolization (here "put" used to indicate the symbolization of the value of the horse). The written calculations could have been done on wooden tablets, covered with wax and written on with styluses. Tablets of this type had been in use since the 12th Century in school activities to write and compose written exercises in prose and verse. Calculations could also be done on paper, which had become increasingly available at the time.

[7] Høyrup (1999) remarked that the Algebra of Master Jacob of Florence (1307) includes colloquial-pedagogical remarks such as "Abiamo dicto de rotti abastanza, però…", "Et se non te paresse tanto chiara questa ragone, si te dico que ogni volta che te fosse data simile ragione, sappi primamente …" "Et abi a mente questa regola", etc.

In this context, the student could hear the teacher's explanation and could see the teacher's gestures as he pointed to the calculations (see Figure 2).

Figure 2. A woodcut showing a teacher examining a pupil (from Orme, 1989, p. 72)

Perhaps, while talking, the teacher wrote something like the text shown if Figure 3.

25	4 I/6	
Ī	I/6	
4		
8 I/	Ī /6	5 /6
8 I/6	5 /6	
$\bar{5}$	49	
Ī	9 4/5	

Figure 3. The teacher's hypothetical written text accompanying the oral explanation (perhaps the written text was less linear than here suggested).

Such a text would support the rich audio (but also perceptual and kinesthetic) mathematical activity that I want to term *oral algebra*. The adjective *oral* stresses the essential nature of the teaching and learning situation –a situation which eventually could also have had recourse to the teacher's notes. In fact, the rich audio and tactile dimension of the learning experience of the time is very well preserved by the look of certain manuscripts. Many of them bear vivid colors and drawings which still stress the emphatic involvement of the face-to-face setting (see Figure 4; for more details, see Shailor, 1994).

As shown by "The gentleman and the servant problem", oral algebra involved making recourse to a text with some algebraic symbolism. However, symbols were not the focus of the mathematical activity. They were part of a larger mathematical discourse, their role being to pinpoint crucial parts of the problem-solving procedure. As we shall see in the next section, at the end of the 15th Century the emergence of printing brought forward new forms of knowledge representation that changed the practice of algebra, as well as the status of symbols.

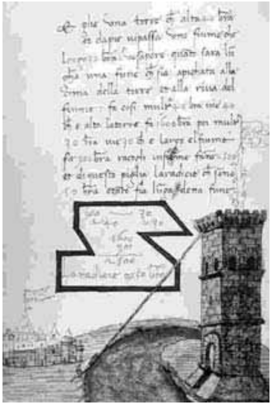


Figure 4. Example of a mathematical manuscript. From Calandri's 15th century *Aritmetica* (Arrighi, ed., 1969, p. 96)

5 Written algebra

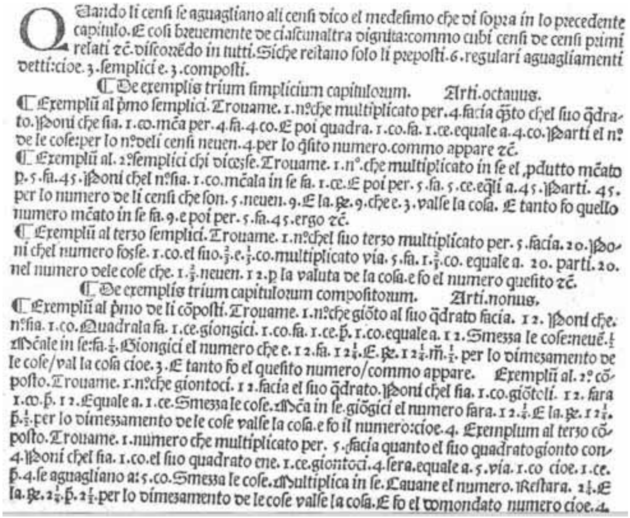
No doubt, the emergence of the printing press not only transformed the forms of knowledge representation, it also altered the classical structures of learned activities. More importantly, the printing press ended up modifying the individual's relationship to knowledge, as is witnessed by the passage quoted in the epigraph of this paper.

With the arrival of the printed book, new cognitive demands arose. The arsenal of resources of oral language, such as vocal inflections, gestures that help to focus the interlocutor's attention on specific points of the problem at hand, the empathy and participation of all the senses, all of this was definitely gone. The reader was left in the company of a cold sequence of printed words. Speech was transformed into writing. And so too was algebra.

For a reader of the 16th Century, to learn algebra from a printed book such as Luca Pacioli's *Summa de Arithmetica geometria Proportioni: et proportionalita* (1494) or Francesco Ghaligai's *Pratica d'Arithmetica* (1521), meant to be able to cope with the enclosed space of the book. It also meant to cope with a mathematical experience organized in a linear way and to overcome the difficulties of a terminology that, for the sake of brevity, used more and more abbreviations, such as "p" for *piu* (plus), "m" for *minus* "R.q." (or sometimes "R") for square root, or contracted words, like "mca" for *multiplica* (multiply) (see Figure 5).

While in a face-to-face interaction ambiguities could be solved by using gestures accompanied by explicative words, the author of the book had to develop new codes to make sure that the ideas were well understood. Syntactic symbols were a later invention to supply the reader with substitutes for the pauses that organize sentences in oral communication[8]. Brackets are perhaps a good example to mention. In a printed book, the numbers affected by the extraction of a square root have to be clearly indicated.

[8] Arrighi tells us that, in his remarkable modern editions of abacists manuscripts, he added modern punctuation (See Arrighi's introduction to his 1970 edition of della Francesca's *Trattato d'Abaco*; see also Arrighi, 1992).



Quando li censi se aguagliano ali censi dico el medesimo che di sopra in lo precedente capitulo. E cosi breuemente de ciascunaltra dignita: commo cubi censi de censi primi relati zc. discorrẽdo in tutti. Siche restano solo li preposti. 6. regulari aguagliamenti detti: cioe. 3. semplici e. 3. composti.

De exemplis trium simplicium capitulorum. Arti. octauus.

Exemplũ al pmo semplici. Trouame. 1. n°che multiplicato per. 4. facia qsto chel suo qdrato. Dconi che sia. 1. co. mcã per. 4. fa. 4. co. E poi quadra. 1. co. fa. 1. ce. equale a. 4. co. Parti el n° de le cose: per lo n° deli censi neuen. 4. per lo qsito numero. commo appare zc.

Exemplũ al. 2° semplici chi dicesse. Trouame. 1. n°. che multiplicato in se el pdutto mcato p. 5. fa. 45. Dconi chel n° sia. 1. co. mcala in se fa. 1. ce. E poi per. 5. fa. 5. ce. eqli a. 45. Parti. 45. per lo numero de li censi che son. 5. neuen. 9. E la. R. 9. che e. 3. valse la cosa. E tanto fo quello numero mcato in se fa. 9. e poi per. 5. fa. 45. ergo zc.

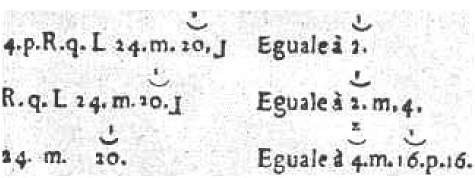
Exemplũ al terzo semplici. Trouame. 1. n° chel suo terzo multiplicato per. 5. facia. 20. Dconi chel numero fosse. 1. co. el suo. ⅓. e. ⅓. co. multiplicato via. 5. fa. 1⅔. co. equale a. 20. parti. 20. nel numero dele cose che. 1⅔. neuen. 12. p la valuta de la cosa. e fo el numero quesito zc.

De exemplis trium capitulorum compositorum. Arti. nonus.

Exemplũ al pmo de li cõposti. Trouame. 1. n° che giõto al suo qdrato facia. 12. Dconi che n° sia. 1. co. Quadrala fa. 1. ce. giongici. 1. co. fa. 1. ce. p. 1. co. equale a. 12. Smezza le cose: neuẽ. ½. Mcale in se: fa. ¼. Giongici el numero che e. 12. fa. 12¼. E. R. 12¼. m. ½. per lo dimezamento de le cose/ val la cosa cioe. 3. E tanto fo el quesito numero/ commo appare. Exemplũ al. 2° cõposto. Trouame. 1. n° che giontoci. 12. facia el suo qdrato. Dconi chel sia. 1. co. giõtoli. 12. fara 1. co. p. 12. Equale a. 1. ce. Smezza le cose. Mca in se. giõgici el numero fara. 12¼. E la. R. 12¼. p. ½. per lo dimezzamento de le cose valse la cosa. e fo il numero: cioe. 4. Exemplum al terzo cõposto. Trouame. 1. numero che multiplicato per. 5. facia quanto el suo quadrato gionto con. 4. Dconi chel sia. 1. co. el suo quadrato ene. 1. ce. giontoci. 4. sera. equale a. 5. via. 1. co cioe. 1. ce. p. 4. se aguagliano a: 5. co. Smezza le cose. Multiplica in se. Cauane el numero. Restara. 2¼. E la. R. 2¼. p. 2½. per lo dimezamento de le cose valse la cosa. E fo el domondato numero cioe. 4.

Figure 5. Excerpt from Pacioli's *Summa d'arithmetica*, edition of 1523

Thus, in his book *L'Algebra*, Bombelli used a kind of "L" and inverted "L" to remove the ambiguity surrounding the numbers affected by the square root sign (see Figure 6).



4.p.R.q.L 24.m.20.⌋ Eguale à 2.

R.q.L 24.m.20.⌋ Eguale à 2.m.4.

24. m. 20. Eguale à 4.m.16.p.16.

$$4+\sqrt{24-20x}=2x$$
$$\sqrt{24-20x}=2x-4$$
$$24-20x=4x^2-16x+16$$

Figure 6. To the left, an extract from *L'Algebra* by Rafaele Bombelli (1572) (Bortolotti, E., (ed.), 1966) with, to the right, its translation into modern symbols. The square root is symbolized by "R.q." ("Radice quadrata"). Parentheses having not yet been invented, to indicate that the square root affects the term 24-20x, Bombelli uses a letter L and the "inverted" letter L

It is clear from the above discussion that the printed book led to a specialization of algebraic symbolism. It conferred an *autonomy* to symbols that they could not reach before. Even if symbols kept the traces of the previous cultural formations where they had played the role of abbreviations, the printed book modified the sensibility of the inquisitive consciousness of the Renaissance. This inquisitive consciousness was now exploring the avenues and potential of the new linear and sequential mathematical experience. Thus, Bombelli's symbolism is made up of abbreviations, but interestingly enough it is also made up of *arbitrary signs*, that is, signs with no clear link to the represented object. Bombelli's representation of the unknown and its powers belong to this kind of sign.

Peletier's algebraic symbolism is also made up of abbreviations (e.g. "R" for *racine*) and arbitrary signs (see Figure 7).

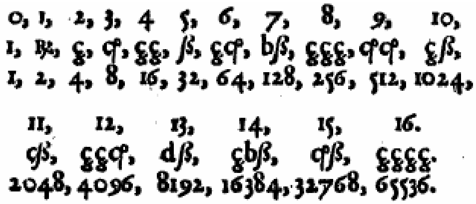
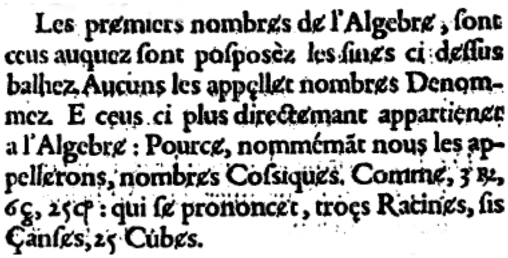


Figure 7. Peletier's symbolism as elaborated in L'Algèbre, 1554, p. 8.

Bombelli's and Peletier's algebraic symbolisms are examples of systems of representation which are partly concrete-contextually based, partly abstract-decontextually based. Their attempts still keep the vestiges of oral algebra, to the extent that when Peletier introduced his abstract symbols, he told his reader how to pronounce them in natural language (see Figure 8).



Les prémiers nombrés dé l'Algebré, sont ceus auquez sont posposèz les sinés ci dessus balhez. Aucuns les appellét nombrés Denommez. E ceus ci plus dirèctémant apartienét a l'Algebré : Pourcé, nommémãt nous les appellérons, nombrés Cosiquès. Commé, 3 ℞, 6 ʓ, 25 cꝑ : qui sé prononcét, troęs Racinés, sis Cansés, 25 Cubés.

Figure 8. Peletier explains how to pronounce the algebraic symbols. *L'Algèbre*, 1554, p. 11.

In light of the previous remarks, can it now be suggested that algebraic symbolism is a corollary of the printing press? My answer is no. The printing press itself was the symptom of a more general cultural phenomenon. It was the symptom of the systematization of human actions though instruments and artefacts. Such a systematization radically modified human experience in the Renaissance, highlighting factors such as repeatability, homogenization and uniformity proper to mass production. As manufacturing, trading, banking and other activities underwent further refinement from the 13th Century onwards, a new crystallization of the economic and conceptual formation of Renaissance culture arose *–efficiency*. Like value, efficiency (understood in its technological sense) became a guiding principle of human activity.

Following this line of thought, in the next section, I will argue in more detail that the changes in modes of representation were not specifically related to printing (which was nonetheless the highest point in the process of the mechanization of all handicrafts), but to the development of a technology that transformed human experience, impressing its mark on the way in which the reflection of the world was made by the inquisitive consciousness of the Renaissance.

6 The cultural and epistemological conditions of algebraic symbolism

Commenting on the differences between the classic geometric procedures ("démonstrations en lignes") and the new symbolic ones, as Bombelli's or Vieta's, Serfati pointed out the huge advantage of the latter in that they bring forward “a strong automatism in the calculations” (Serfati, 1999, p. 153).

A similar remark was made by Cifoletti in her studies on Peletier. She rightly observed that Peletier’s

> principal innovation resides in the introduction of as many symbols as there are unknowns in the problem, as well as in the fact that the unknowns in the problem correspond to the unknowns in the equations, in contrast to what was being suggested by, for example, Cardan and Stifel. (Cifoletti, 1995, p.1396)[9]

The introduction of arbitrary representations for the several unknowns in a problem is indeed part of Peletier’s central idea of elaborating an “automatic procedure” (Cifoletti, 1995, pp. 1395-96; Cifoletti, 1992, p. 117 ff.) to tackle the problems under consideration. Instead of having recourse to sophisticated artifices like those used by Diophantus several centuries before the Renaissance, the symbolic representation of several unknowns offered the basis for a clear and efficient method.

Clarity and efficiency of method, of course, are cultural concepts. Diophantus would have argued that his methods were perfectly clear and efficient (see Lizcano, 1993). And Plato would have claimed that efficiency (in its technological sense) should be the last of our worries[10].

Thus, the emergence of algebraic symbolism appears to be related to a profound change around the idea of *method*. Jacob Klein clearly noticed this when he stated that what distinguishes the Greek algebraists, like Diophantus, from the Renaissance ones is a shift from *object* to *method*: ancient mathematics

> […] was centered on questions concerning the mode of being of mathematical objects […]. In contrast to this, modern mathematics [i.e. 16th and 17th Century mathematics] turns its attention first and last to *method as such*. It determines its objects *by reflecting on the way in which these objects become accessible through a general method*. (Klein, 1968, p. 122-123; emphasis as in the original)

The difference between “ancients” and “moderns” can be explained through an epistemological shift that occurred in the post-feudal period. Referring to 16th Century “modern” epistemology, Hanna Arendt argues that the focus changed from the object to be known to the process of knowing it. Even if “man is unable to recognize the given world which he has not made himself, he nevertheless must be capable of knowing at least what he has made himself.” (Arendt, 1958a, p. 584). Or “man can only know what he has made himself, insofar as this assumption in turn implies that I ‘know’ a thing whenever I understand how it has come into being”. (*op. cit*. p. 585; the idea is elaborated further in Arendt, 1958b).

The use of letters in algebra, I want to suggest, was related to the idea of rendering the algebraic methods efficient in the previous sense, that is to say, in accordance to the general 16th century understanding of what it means for a method to be clear and systematic, an understanding that rested on the idea of efficiency in the technological sense. You write down your unknowns, and then you translate your word-problem. Now you no longer have words with meanings in front

[9] “L’innovation principale réside dans l’introduction d’autant de symboles qu’il y a d’inconnues dans le problème, et en ce que les inconnues du problème coïncident avec les inconnues des équations, contrairement à ce que suggéraient, par exemple, Cardan et Stifel.″ (Cifoletti, 1995, p. 1396)]

[10] The use of mechanical instruments made by e.g. Eudoxus and Architas was indeed criticized by Plato: “But Plato took offense and contended with them that they were destroying and corrupting the good of geometry, so that it was slipping away from incorporeal and intelligible things towards perceptible ones and beyond this was using bodies requiring much wearisome manufacture.” (Plutarch, Lives: Marcellus, xiv; quoted by Knorr 1986, p. 3).

of you. What you have is a series of signs that you can manipulate, in a machine-like manner, in an efficient way. Signs become manipulated as commodities were manipulated in the 16th century market place. And as you do not even need to know who made the commodity, in the same way you do not need to know what objects the signs refer to. We are here in front of a new epistemological stratum that regulates in a same way the abstraction of the referent in algebra and in the economic world.

In more general terms, what I want to suggest is that the social activities of the post-feudal period were highly characterized by the two crystallizations of the economic and conceptual formations of Renaissance culture discussed in this paper, namely *value* and *efficiency*. Mathematical thinking as a reflection of the world was shaped by these crystallizations. These crystallizations led to two points. On the one hand, to an unprecedented creation of instruments –e.g. military machinery, da Vinci's impressive investigations on flying machines, parabolic mirrors, pulleys, etc. (see Pedretti, 1999), Dürer's perspectograph, and so on. On the other hand, to a reconceptualization of mathematical methods and the creation of new ones (e.g. analytic geometry) modelled to an important extent on the technological metaphor of efficiency.

Within this context, the effort carried out by one of the fathers of algebraic symbolism to legitimize the use of instruments in mathematics is fully understandable. Indeed, in his *Geometry*, Descartes (see Figure 9) complains about the lack of interest shown by ancient mathematicians for "mechanical curves", i.e. curves constructed with some sort of instruments for, as he argues, one must to be consistent and then also reject circles and straight lines, given that they are constructed with rule and compass, which are instruments too (Descartes, 1637/1954, pp.40-43; see Figure 9):

To sum up, although certainly not the only elements, value and efficiency (in its technological sense) helped to build the epistemological foundations for the emergence of algebraic symbolism.

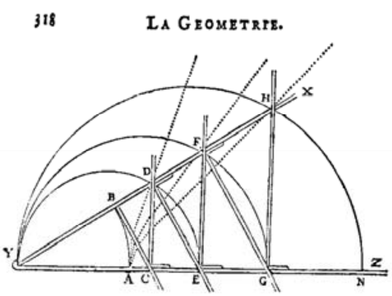


Figure 9. Descartes' construction of a curve with the help of an instrument made up of several rules hinged together. Descartes argued that curves described by several successive motions or continuous motion of instruments may yield exact knowledge of the resulting curve (Dover edition of *La Géométrie*, 1954, p. 46).

7 Synthesis and Concluding Remarks

Cultural conceptual categories are crystallizations of historic, economic and intellectual formations. They constitute a powerful background embodying individuals' reflections of the world as it appears to them, for living in a culture means to be diversely engaged in the interactive zones of human activity that compose that culture.

The two aforementioned crystallisations were instrumental in creating the conditions for a new kind of inquisitive consciousness –a consciousness which expressed its reflection about the world in terms of systematic and efficient procedures.

That the previous crystallizations reappeared in other sectors of human life can indeed be seen if we turn to painting. Perspective calls for a fixed point of view, an enclosed space, much like the page of the written book. It supposes homogeneity, uniformity and repeatability as key elements of a world that aligns itself according to the empire of linear vision and self-contained meaning (see Figure 10).

Perspective is a 'clear method' with which to represent space in a systematic and efficient instrumental form (see Figure 11), in the same manner that the emergent algebraic symbolism is a 'clear method' with which to represent word-problems through symbols. Symbolic algebra and perspective painting in fact obey the same form of cultural signification. This is why perspective lines are to the represented space what algebraic symbols are to the represented word-problem.

Figure 10. A perspective drawing from 1545

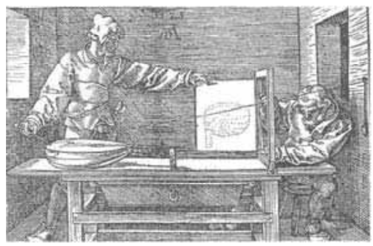


Figure 11. Dürer's perspectograph or instrument to draw and object in perspective

It is important to note at this point in our discussion that the two aforementioned crystallizations, value and efficiency, were translated in the course of the activities into an ontological principle which, during the Renaissance, made the world appear to be something homogeneous and quantifiable in a manner that was unthinkable before. Converted into an ontological principle, it permeated the various spheres of human activity. In the sciences, it led to a mechanical vision of the world. In mathematics, such a principle, which nonetheless remained implicit, allowed Tartaglia, for instance, to calculate with what would have been considered non-homogeneous measures for the Greek episteme. As Hadden, remarked,

> Niccolo Tartaglia (d. 1557), for example, formulates a statics problem in which it is required to calculate the weight of a body, suspended from the end of a beam, needed to keep the beam horizontal. Tartaglia's solution requires the multiplication and division of feet and pounds in the same expression. Euclidean propositions are employed in the technique of solution, but Euclidean principles are also thereby violated. (Hadden, 1994, p. 64)

The homogeneous and quantifiable outlook of things (see Crosby, 1997) was to the ontology of the Renaissance what the principle of non-contradiction was to Greek ontology or what the yin-yang principle of opposites was to the Chinese one.

It is perhaps impossible to answer, in a definitive way, the question of whether or not the alphanumeric algebraic symbolism of today could have emerged had printing not been invented. Piero della Francesca's timid algebraic symbolism suggests, however, that the idea was 'in the air'

– or to say it in more technical and precise terms, the idea was in the *zone of proximal development* of the culture[11]. Perhaps printing was a catalyzer that helped the Renaissance inquisitive consciousness to sharpen the semiotic forms of knowledge representation in a world that substantially transformed human experience by the use of artifacts and machines and which offered a homogeneous outlook of commensurate commodities through the cultural abstract concept of value. Value has certainly shown that things are interchangeable and that their representation is in no way an absolute claim for the legitimacy of the represented thing. Giotto's paintings are representations in this modern sense of the word: they do not claim a coincidence between the representation and the represented object. Stories, in Giotto's paintings, are often told by moving a few signs around the painting surface (the rock, the dome, the tree, the temple, the heritage, the church, etc.), much as algebraic symbolism produces different stories by moving its signs around.

Peletier's immense genius led him to see that the key concept of our contemporary school algebra is the equation. For sure, Arab algebraists classified equations before abacists such as Pacioli or della Francesca and Humanists like Peletier or Gosselin, but these equations referred to 'cases', distinguished according to the objects related by the equality. For Peletier, the equation belongs to the realm of the representation: an equation is an equality, not between the objects themselves, but as they are *dénommés,* that is, *designated* (see Figure 12).

For Peletier, the equation is a semiotic object. Peletier belongs to the post-feudal ear, the era where, as Foucault (1966) remarked, things and names part company[12].Value, as a cultural abstract concept, has made the place of things in the world relative, thereby leading to new forms of semiotic activity.

As Otte (1998, p. 429) suggested, the main epistemological problem of mathematics lies in our understanding of 'A=B', that is, in the way in which the same object can be diversely represented[13]. Abacists were the first to tackle this problem through the intensive use of the cultural category of value, thereby opening the door for subsequent theorizations, as the mathematician Bochner very well realized, although not without some surprise. He said:

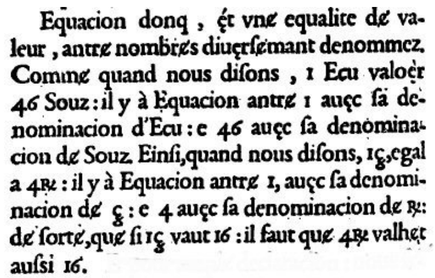

Equacion donq, ét vne equalite de valeur, antre nombres diuerſemant denommez. Comme quand nous diſons, 1 Ecu valoet 46 Souz: il y à Equacion antre 1 auec ſa denominacion d'Ecu: e 46 auec ſa denominacion de Souz. Einſi, quand nous diſons, 1ç, egal a 4R: il y à Equacion antre 1, auec ſa denominacion de ç: e 4 auec ſa denominacion de R: de ſorte, que ſi 1ç vaut 16: il faut que 4R valhet auſsi 16.

Figure 12. Peletier's definition of equation. *L'Algèbre*, 1554, p. 22

> It may be strange, and even painful, to contemplate that our present-day mathematics, which is beginning to control even the minutest distances between elementary particles and the intergalactic vastness of the universe, owes its origination to countinghouse needs of 'money changers' of Lombardy and the Levant. (Bochner, 1966, p. 113)

Perhaps our debt to the abacisits would be less painfully resented if it were recognized that knowledge relates to culture in the precise sense that the activity from which the object of

[11] The concept of *zone of proximal development* was introduced by Vygotsky (1962) to explain the ontogenesis of concepts in individuals. I am expanding it here to account for that which becomes potentially thinkable and achievable in a culture at a certain moment of its conceptual development.

[12] See also Nicolle, 1997.

[13] See also Otte (in press).

knowledge is generated impresses in the object of knowledge the traces of the conceptual and social categories that it mobilizes, and that what we know today and the way that we have come to know it bear the traces of previous historical and cultural formations.

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PANEL: PROOF IN HISTORY AND IN THE CLASSROOM

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The causality proof scheme[1]

"We do not think we understand something until we have grasped the why of it. … To grasp the why of a thing is to grasp its primary cause," asserts Aristotle in Posterior Analytics. Some 16-17th Century philosophers argued that mathematics is not a perfect science because "implication" in mathematics is a mere logical consequence rather than a demonstration of the *cause* of the conclusion. If we are to draw a parallel between the individual's epistemology of mathematics and that of the community, the following questions are of paramount importance: Was the causality issue of marginal concern to the mathematics of the sixteen and seventeen centuries, or had it significantly affected it? To what extent did the practice of mathematics in the sixteen and seventeen centuries reflects global epistemological positions that can be traced back to Aristotle's specifications for perfect science? Mancosu (1996) argues that the practice of Cavalieri, Guldin, Descartes, and Wallis, and other important mathematicians reflects a deep concern with these issues. He shows, for example, how two of the major works of the 1600s—the work by Cavalieri on indivisibles and that by Guldin, his rival, on centers of gravity—aimed at developing mathematics by means of direct proofs. These two mathematicians, argued Mancosu, explicitly avoided proofs by contradiction in order to conform to the Aristotelian position on what constitutes perfect science—a position Aristotle articulated in his Posterior Analytics. Mancosu (1996) also argues convincingly that Descartes, whose work represents the most important event in seventeenth-century mathematics, was heavily influenced by these developments. Descartes appealed to a priori proofs against proofs by contradiction because they show how the result is obtained and why it holds, and they are *causal* and ostensive.

The history of the development of the concept of proof may suggest that our current understanding of proof was born out of an intellectual struggle during the Renaissance about the nature of proof—a struggle in which Aristotelian causality seem to have played a significant role. If the epistemology of the individual mirrors that of the community, we should expect the development of students' conception of proof to include some of the major obstacles encountered by the mathematics community through history. We conjecture that Aristotelian causality is one of these obstacles. In my studies, causality has been observed with able students, who seek to understand phenomena in depth, than with weak students who usually are satisfied with whatever the teacher presents.

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[1] "Proof scheme" is the sense given in Harel & Sowder (1998).

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Proof in History and the Classroom

Historically the philosopher Thales has been accredited as the inventor of the mathematical proof. I have seen an argument which questions this honour where the main point is that since Thales did not have an axiom system, he could not prove anything at all.

However, the purpose of a proof is to convince an audience, by making them "see for themselves" that what I say is true. /What is needed is not a system of axioms, but that the prover and the audience agree on what is considered as known and what is accepted as obvious or convincing./ Therefore it can be said that **in** this respect, Thales was in a situation similar to that of a school teacher in front of a class.

Ever since the time of Thales, the "mathematical proof" has been the distinguishing feature of mathematics - nevertheless as Lakatos observed in his famous Proofs and Refutations, "yesterday's proof" might be just a good joke today.

Therefore a teacher who wants to convey the spirit of mathematics to her students has to create an understanding of what a mathematical proof is, and hopefully also a feeling for it. Mathematics is often considered as an authoritarian subject at school, while it could in fact be the least authoritarian, and thereby the most democratic subject of all. When a student has understood a proof, she knows that what the teacher told is true - not because the teacher said so, but because she has understood the proof.

The issue of using proof in the class-room is certainly one of the most important questions to discuss among all teachers of mathematics. It was therefore clear to us that we needed a Panel Discussion concerning proofs, and at HPM such a discussion should consider both the historic and the educational aspect of this issue. The participants of this discussion were Guershon Harel, Siu Man-Keung, Tasos Patronis and Anders Öberg, with me as coordinator. Unfortunately the first edition of the proceedings was published in such a haste that only Guershon Hare's contribution was included.

I shall conclude this posteriorly written introduction to the panel discussion by telling about my own favorite “first proof in class”. I actually believe that this proof can be given already in primary school, perhaps in the second or third grade. The proof is preceded by asking the students to make a simple drawing on paper as follows.

The teacher starts by asking the students to put say 7 dots on a piece of paper, and then connect pairs of points by drawing a curve between them. The rules are that every curve has to go between two different dots, and there is only allowed one curve between any two given dots. It is of course not necessary to connect all pairs of points. Two points are then said to be neighbors if there is a curve between them. The teacher then promises any student who is able to draw curves in such a way that all points have a different number of neighbors will be given something - say a small amount of money.

When the students have tried for some time, somebody will probably ask if it is possible, and it is then time to have a vote on whether it is possible or not. Perhaps it is then time to tell that it is impossible and hope for the question - how do we know that?

One can then look at the simpler cases, 2 points, 3 points, 4 points, before one goes to the general case, and introduces the pigeon-hole principle.

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Geometric explanation in elementary number theory from Pythagorean tradition to students of today: the case of triangular numbers

According to the modern Greek historian of Mathematics Evangelos Stamatis (1898-1990), Triangular and Polygonal Numbers were constructed within the Pythagorean Tradition (as it appears in Nicomachus' *Introduction to Arithmetic*) inductively, starting from a *unit* (*monas*), which was given a particular polygonal shape. This unit was considered as a "potential" triangle, square or other regular polygon, which then was successively "augmented" into a similar polygon of sides 2,3,4 etc. by adding, each time, a suitable *gnomon,* i.e. a shape representing the difference between two successive polygonal numbers. This inductive construction can explain several properties of such numbers, as e.g. that the n^{th} square number n^2 is the sum of all n first odd numbers $1+3+5+\ldots+(2n-1)$. However, it seems that it is not possible to use *gnomons* directly in order to find a "closed" form for the computation of the n^{th} triangular number

$$T_n = 1+2+3+\ldots+n$$

The problem of computing T_n in an easy way (and its solution) was published together with Nicomachus' *Introduction to Arithmetic* by R. Hoche in 1866, but apparently this problem does not belong to the work of Nicomachus. Modern textbooks of Elementary Number Theory sometimes re-arrange T_n into an orthogonal triangle shape, which they complete to a square or a rectangle and then compute T_n as the number of lattice points belonging to half of this rectangle (figures will be used in my panel 10-minutes introduction of the subject). Now this switch of the shape of representation, from a "regular" to an orthogonal one, causes some unexpected and interesting confusions to students of today and reveals a notable inherent logical difficulty in geometric explanations of this kind.

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Proof in History and in the Classroom

Through examples this introductory talk tries to explore the practice of mathematical pursuit, in particular on the notion of proof, in a cultural, socio-political and intellectual context. Not so much attention would be paid to the evolution of the standard of rigour or to the epistemological aspect of a mathematical proof (like in *Proofs and Refutations* by Imre Lakatos). Because of time constraint not much attention would be paid to the technical detail of a proof of a specific theorem either. Rather, we try to look at a few examples, including:

(1) the influence of the exploratory and venturesome spirit during the ‘era of exploration’ in the 15th and 16th centuries on the development of mathematical practice in Europe,
(2) the influence of the intellectual milieu in the period of the Three Kingdoms and the Wei-Jin Dynasties (in the 3rd and 4th centuries) in China on mathematical pursuit as exemplified in the work of LIU Hui,
(3) the influence of Daoism in mathematical pursuit in ancient China with examples on astronomical measurement and surveying from a distance.

One objective in mind in the discussion is to show how mathematics constitutes a part of human endeavour rather than stands on its own as a technical subject, as it is commonly taught in the classroom. The examples may also suggest ways to enhance understanding of specific topics in the classroom, but that would be best left to those who are doing the actual teaching in the classroom. Comments and suggestions are most welcome during the open discussion.