ON DIALECTICAL RELATIONSHIPS BETWEEN SIGNS AND ALGEBRAIC IDEAS[1]

Luis Radford
Université Laurentienne
Ontario, Canada

Monique Grenier
Conseil de l'éducation de Sudbury
Ontario, Canada

The goal of this paper is to provide some preliminary elements for a discussion about the relationships between signs and algebraic ideas. In order to do so, in section 1, we discuss some semiotic and philosophical ideas about signs and symbols. The ideas drawn in section 1 give us a new perspective to understand some elements of the epistemological dialectic between signs and algebraic ideas (section 2). In section 3, we present some data obtained from a teaching sequence, shaped by our epistemological analysis, whose aim was to help students to evolve towards abstract symbols-ideas levels.

1. Signs, Icons and Symbols

While it seems that there is a general consensus considering symbols as a driving force of algebraic thinking, it is much less obvious to say how symbols can be used to promote algebraic ideas. Of course, we may say that algebraic ideas are actually promoted when students manipulate symbols like x, y, z. However, as a closer look at the problem shows, the modern scientific culture on which such a belief may be founded cannot suffice to sustain the thesis that algebraic ideas are automatically lagged behind symbols. Thus, in order to approach the question of how can symbols be used to promote ideas, we first need to examine, to some extent, what a symbol is.

Symbol is not seen as synonymous with sign. Let us start with the latter. What is a sign? Many linguists agree in saying that a sign is something used everyday to communicate and to signify. However, the functional status of signs do not characterize them. Mediaeval scholars used to say that a sign is something which is placed instead of something else (*aliquid stat pro aliquo*). Following this tradition, C. S. Peirce –who determinately influenced the semiotic research of the 20th century– defined, at the end of the last century, a sign as "something which stands to somebody for something in some respect or capacity" (Buchler, 1955, p. 99).

According to Peirce, a sign captures only an aspect of its *object*, this aspect is the *ground* of the sign, that is, a component of the signified (*signifié*) associated to the *object* (see Eco, 1985, pp. 34-35). On the other hand, a sign may refer to another sign by virtue of a certain abstraction. A sign, Peirce says, "creates in the mind of the person an equivalent sign, or perhaps a more developed sign" (*op. cit.* p. 99) and he calls this last sign the *interpretant* of the first sign. The *interpretant* is a new sign (see fig. 1) containing a richer cognitive content (Eco, 1988, pp. 153-54).

Peirce divided the signs into three categories that have been adopted by modern semiotics and are nowadays, as Eco notes (1992, p. 11), of universal usage. The three categories are: *Index*,

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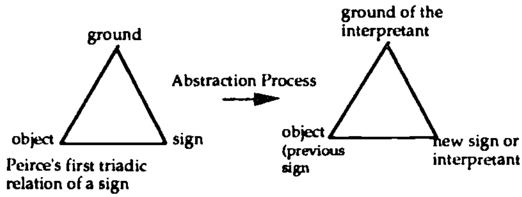
Icons and *Symbols*. For our purposes, we only need to consider the last two. According to Peirce, "an *Icon* is a sign which refers to the object that it denotes merely by virtue of characters of its own, and which it possesses, just the same, whether any such object actually exists or not" (Buchler, 1955, p.102). The Icon reflects, then, some 'resemblance' (physically or other) with its object. In contrast, the *Symbol* is considered by Peirce as an arbitrary sign related to its object by virtue of a law or convention.

From an educational point of view, we are interested in elucidating the signifying function of symbols –a problem related to the question that drives our research and that we raised before, that is, how symbols can promote ideas. In particular, we are interested in the following set of questions which we will call Question (α):



recurrent semiotic relationship between signs

(α_1)Let s_1 be a sign of content or ground g_1 that evolves into another sign s_2 of a richer content g_2. Is the produced change caused by a modification of the first sign or by a modification of the first content? Let θ be the starting term (thus, $\theta \in \{s_1, g_1\}$).

(α_2) How does the changing process take place? More specifically:

- ($\alpha_{2.1}$) (*external agent*) What is it that makes θ change?
- ($\alpha_{2.2}$) (*internal dynamic*) How do the different components s_1, g_1, s_2, g_2 interact between themselves during the changing process?

The answer to this question depends on the content related to the ground and on some cultural aspects in which the signifying act takes place. The answers will also depend on what we can call the 'subject's experiential field' related to the object or concept (a field that recovers the encounters and experiences between the subject and the concept). Here, we do not need to consider this question in all of its generality; we shall circumscribe it to the case of algebra. In order to pursue our investigation let us turn our attention to the philosophical perspective about signs (something that, in contrast to semioticians, they call symbols). Even though many philosophers may agree in considering that a symbol is something placed instead of something else, as the mediaeval tradition did, the role that they give to symbols is that of accomplishing a kind of privileged transcendental 'contact' with the idea (or the *object*, to use the Peircean term) that the symbol is trying to catch (see Durand, 1964). The 'power' of symbols is precisely to allow us to make *definable* the *undefinable*, to express beyond words that which is essentially inexpressible and to «translate», beyond perceptible forms, that which is absolutely «undefinable» (Juszezak, 1985, p. 8). The symbol becomes the *epiphany* (that is, the apparition) of the unfigurable object. In the case of emerging concepts, the process of symbolization becomes an intellectually difficult adventure. In fact, how does one *call*[2] or *name* the object that does not yet have an intelligible form[3]?

To sum up our discussion, Peirce's approach to the concept of sign allowed us to raise the Question (α) that we consider relevant to teaching. The philosophical approach to the problem of symbolization sketched above makes it possible to go a step further by providing a different perspective to see the nature of the links between a *sign* and its *object* (or *idea*). The philosophical approach may not satisfactorily answer our questions insofar as it may persist in considering the *object* as an external object with a somewhat independent life from the cognizer. However, both approaches provide us a starting point to consider our question (α) from an educational perspective.

2. Some Epistemological Elements of Algebraic Signs: Naming the Un-nameable

In this section, we want to examine briefly some elements of the epistemological dialectic between signs and algebraic ideas, focusing our attention on the concept of *unknown*. In order to do so, let us remember that Babylonian scribes developed problems about geometrical figures (squares, rectangles and so on); in many of these problems, the unknown was referred to by its material name: e. g. a width, a length. No specific name was then created to designate the emerging concept. Furthermore, according to Høyrup's recent historical reconstruction (Høyrup, 1990), problem-solving procedures for many problems were guided by *figures* representing the problem (e.g. squares, rectangles) from which some parts were cut and then transferred and pasted to other sides of the figure, while other figures were added, when necessary, in order to reach a final square. The point that we need to stress is that the algebraic thinking underlying the solution of such problems was essentially *iconic*.

In contrast, Western mediaeval mathematicians used the latin word *res* and later the italian word *cosa* (the *thing*) to represent the concept of unknown. Thus, there was a specific name for the concept of unknown that applied to a great variety of problems. Even though the word *res* was taken, for a time, as synonymous with *radice* (root) -which has an obvious geometric sense- *res* and *thing* later acquired a contextual autonomy, thus becoming a *symbol* (in Peirce's sense).

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[2] It is worthwhile to note that one of the meanings of the verb /to call/ is «to demand or ask for the presence of». Another one, even more suggestive, is «to invoke solemnly». To call, then, allows one to make something *appear*, to become intelligible to our intellect. In this context, the transcendental contact with the unattainable thing will be ensured by the symbol.

[3] Semioticians did not miss this point. For instance, in a recent book, Eco says:

"... one cannot have at one's disposal the appropriate expression until one has differentiated the content system to an appropriate degree. It is a paradoxal situation whereby the expression must be established on the basis of a non-existant content model before it can be expressed in some manner. The producer of the signs has a very clear idea of *what* he would like to say, but does not know *how* to say it..." (Eco, 1992, p. 30)

Because of the fact that the name of the concept of unknown was no longer sufficient to explain itself (in contrast to the case of icons that are self-sufficient), we find, in the 14th century, the virtuoso Maestro A. de Mazzinghi explaining the thing as "an occult or hidden quantity" –that is, a quantity whose identity will be discovered through the problem-solving process.

It is extremely difficult to retrace the conceptual movement of signs and their corresponding grounds. However, according to our research, we can suggest that the conceptual movement leading to the symbol /thing/ in the 14th century was strongly supported by an enlargement of the type of problems to solve. The square and its root did not apply to extra-geometric problems. Thus, one could be lead to extend the geometrical ideas (based on the measure of segments and surfaces) and to start considering them in a unifying numerical perspective which lead to new ideas and signs. In terms of our Question (α), the enlargement of the type of problems would play the role of an external releasing agent of the change process. The root -an iconic sign- was absorbed by an existing symbol -the thing- and this process was accompanied by an absorption and restructuration of ideas (see $\alpha_{2.2}$).

Res and *thing* (as well as the Arabian word *shay'* for the concept of unknown) were a *catalyst* in the development of algebraic symbols. We find, in the 15th and 16th centuries, different mathematicians engaged in the search of *shorter* symbolic representations –a search mainly promoted by the need to find an easier way to carry out calculations which had become very encumbering when problems became more complex. One of these attempts was made by Piero della Francesca (*fl.* 1450); however, because of its geometric connotation, his sign system does not reach the point of independence between content and expression. Della Francesca's system can only be seen as an iconic one (see figure 2)[4]. In contrast, Michael Stifel (1544) developed a symbolic system. However, his system had the inconvenience of making it impossible to represent the operational links between the unknown and its powers (e. g. the square of the unknown is represented by a completely independent sign from the unknown, see figure 2).

Attempts were also made by other mathematicians (e.g. Bombelli). The system that finally prevailed was the one introduced by Viète and reverted to and improved by Descartes. As Cajori (1919) says, Descartes' choice was arbitrary. And following Peirce, it is the condition for a system to be called symbolic. With the advent of the socially accepted Descartian symbolic system, signs moved into another status: symbols became genuine

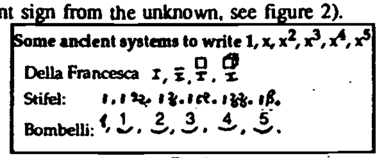


Fig. 2

[4] It is very interesting to note that della Francesca's efforts to construct a shorter symbolic system lead him to an iconic one, which can be seen as taking a step backwards. This failure may be seen as a symptom of the difficulties that one encounters when one tries to reach more abstract levels of thinking.

mathematical objects. The rupture with the mediaeval tradition can be seen through the fact that before algebra reached its symbolic new realm, it was not possible to pose problems using symbols only. Problems were posed in verbal form; signs were then used to translate and solve them. The recognition of the status of mathematical objects for symbols made it possible to pose problems within the symbolic system. As we can expect, the rupture with the symbolic mediaeval tradition of *res* and *thing* raised new problems. Indeed, the ground of a sign was no longer an *idea* (in its trivial sense) but a *symbol*. The recurrent chain of signs dealt with *symbolic interpretants*. As a consequence, a different way of mathematical thinking was developed –a symbolic algebraic thinking.

3. From Iconic to Symbolic Algebraic Thinking: An Example

We are now going to present some experimental results of a teaching sequence for the introduction of algebra. The introduction of algebra has been studied intensively over the last years (see e.g. Rocha Falcão, 1995; Arzarello *et al.*, 1994; Bednarz and Janvier, 1994; Filloy and Rojano, 1989, Herscovics and Kieran, 1980). There are even some commercially registered manipulatives (e.g. Hands-on Equations® and Alge-Tiles™). A difference between the previous approaches and ours is to be found in the historico-epistemological basis underlying our teaching sequence. This basis allowed us to formulate our objective and to structure the sequence as follows. We wanted to help students to evolve from concrete to abstract symbols-ideas levels through a problem-solving process based on two main basic historical algebraic principles underlying the transformation of equations: (a) a *conservation–equality principle* that allows one to carry out calculations with the constant known terms; this principle also allows one to increase or decrease, in the same proportion, both members of an equation –therefore, to find an equivalent equation, 'proportional' to the preceeding equation– (see Radford, 1995a, pp. 79-80) and (b) a *restoration principle* (something historically called the rule of *al-gabr*, related to subtractive operations) allowing one to restore or fix an 'uncompleted' or 'broken' expression (see Radford, 1995b, pp. 31-32).

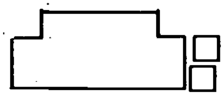
We carried out a three-step teaching sequence in which students had to solve some word-problems using (*i*) manipulatives, (*ii*) icons and (*iii*) symbols.

Our teaching sequence (that was video-taped) was first experienced with 6 students ,14-16 years old, from a resource center for students having difficulties and later with a regular class of Grade 9 students in a secondary school in Ontario. Working in cooperative groups of 3, they were asked to answer some word-problems classified into 6 categories. Here, we report some results from only two of them. One such category was the 'hockey card problems' (a category that can be ackwardly modelled by equations of the type $a_1x + a_2x = a_3x + a_4$ $(a_i \in N^*)$(see problem 1) and which was preceeded by another category of problems (bag-problems) related to equations of the type $a_1x + a_2 = a_3$ $(a_i \in Q^*)$. The other category that we shall consider here was the pizza-problems –a category related to equations of the type $a_1(x-a_2)+a_3=a_4(x-a_5)+a_6$ $(a_i \in N)$ (see problem 2).

Problem 1:
I have 3 envelopes each containing the same number of hockey cards plus 4 extra loose cards. I give Jacques 2 envelopes plus 1 extra card and I give Paul 1 envelope plus 3 extra cards. If I gave Jacques and Paul the same number of hockey cards, how many hockey cards are there in one envelope?

Problem 2:
André must purchase the same number of pizza slices as Louise. When they arrive at the pizzeria, they realize that the pizzas are all missing 2 slices. André buys 3 incomplete pizzas while Louise buys 1 incomplete pizza plus 4 extra slices.How many slices does one complete pizza have?

In step (*i*), the students were provided with concrete material that we designed according to the problems. For instance, for the pizza-problems, we gave each group of students a kit containing cardboard pizzas as shown in fig. 3.



A pizza with two missing slices
figure 3

Before solving the problems, the students were familiarized with principle (a) which acquired a concrete meaning in terms of the two-plates balance that stand for the equality of expressions. Principle (b) was not directly taught. In step (*ii*) students no longer had, at their disposal, the manipulatives; they were asked to make the designs to solve the problems. The designs correspond to (perceptual) icons (see fig. 4).

The students solved, without difficulties, the problems of steps (*i*) and (*ii*). During the problem-solving process, they referred their actions to the algebraic rules (a) and (b).

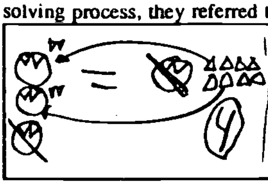


figure 4 (problem 2)

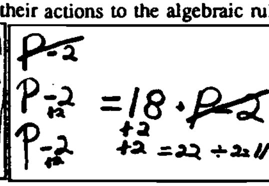


figure 5

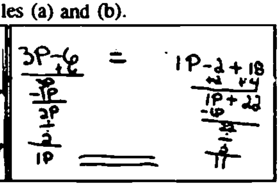


figure 6

Some objects of many problems in step (*ii*) were chosen in such a way that they were too long to design (e.g. 21 hockey cards). Thus, instead of drawing the objects, we asked the students to use numbers and letters. Spontaneously they used the first letter of the word. Thus, for example, pizzas were represented by the letter p, while envelopes were represented by e. Their choices coincide with a pattern mentioned by Ard 1989, p. 257:

> "Many, if not most, symbols were originally. suggestive of a letter or sound. It is not accidental that f is the most common symbol for a function and $\vec{v}$ is the most common symbol for a vector".

In terms of the resolution of problems, the passage from step (*ii*) to (*iii*) was succesfully done. Of the group of 6 students, all, except one (to whom we shall return later; fig. 7), were able to deal with problems placed in the 'other side' of the «didactic cut» (Filloy and Rojano, 1989).

From (*ii*) to (*iii*), students kept using the *algebraic* problem-solving procedures (based on rules a and b) that they developed cooperatively in steps (*i*) and (*ii*).
Concerning the dialectic between signs and ideas, something interesting to note in step (*iii*) is the trace of (perceptual) iconic algebraic thinking on symbolic algebraic thinking. According to principles (a) and (b), to solve pizza-problems, the students eliminated known and unknown similar terms and completed the remaining uncompleted pizzas. In (*iii*), as it can be seen in fig. 5 (which refers to a problem of equation $3(p-2)=18+(p-2)$, the students eliminated first the term /p-2/ that designates a pizza with two missing slices; then they completed the two remaining uncompleted pizzas, taking care (according to principle a) to add the 2 slices twice to the member on the right of the equation (something that is placed exactly under the number 18 that represents eighteen slices of pizza). The symbol /p-2/ is seen in a *synthetic* way rather than in an *analytical* one. This means, in terms of our Question (α) (section 1), that, here, we are dealing with a phenomenon in which the sign effectively changes (from icon to symbol) while –as far as we can see– its ground did not experience a comparably significant change.
However, we were able to observe, in another instance of our teaching sequence, students moving more radically to a more abstract signs-ideas level. In fact, when solving the aforementioned pizza problem (fig. 5), a student wrote the equation /3p-6=1p-2+18/ (see fig. 6) which witnesses a first spontaneous attempt to evolve to a more abstract way of thinking. Indeed, the symbol /3p-12/ does not refer to any concrete data; it can only be understood in terms of an abstract grouping or association of the given data; in doing so, new *interpretants* (in a Peircean sense) of previous signs were constructed.
Let us now discuss one of the problems that we could detect in the passage from (ii) to (iii). Icons give a somewhat tangible spacial presence to the objects that they are representing. Thus, when students applied principles a and b, the transformed equations appeared *naturally*: the resulting equations were *there*. There was no difficulty in giving the equation any specific *spacial* configuration (see fig. 4). In contrast, when students had to use symbols, after a transformation, they had to represent the new equation (which reflects the current state of the problem-solving procedure). The new equation requires keeping track of the results of the transformation actions as well as co-ordinating, in a more abstract way, the different relationships between all of its terms. This requirement (upon which is largely based the success of a symbolic algebraic thinking) is not simple, as we can observe. In fact, fig. 7 shows the case of a student that solved completely similar problems in steps (i) and (ii) but was unable to move successfully to a symbolic algebraic thinking.

The video analysis revealed that the student loses sense of the actions undertaken. The underlying algebraic equality, which was the axis of the procedure, is lost. She is not able to give a sense to the symbols /1e+3/ (i.e. the transformed member on the left) and /5/ (i.e. the transformed member on the right).

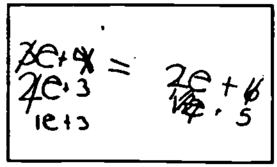


figure 7

To conclude our discussion, let us note that, in step (*iii*), the students conceived the written equation as a static support upon which one carries out the actions in order to solve the problem. This means that the written equation does not evolve in a sequential manner, line by line, as one would expect in the case of a 'competent' utilization of the algebraic language. Here, the written equation has a *heuristic* value that guides the actions of the resolution. The 'competent' utilization of the equation rests upon a code that is, like all other codes, socially constructed. It seems to us that it should be from this viewpoint that the transformations of equations be approached in the classroom. However, only after the students have had the opportunity to construct a content (i.e. a meaning) for the symbols. Otherwise, the symbols remain empty (i.e. meaningless) and, strictly speaking, cannot be symbols.

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