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CHAPTER 7

SOME REFLECTIONS ON TEACHING ALGEBRA THROUGH GENERALIZATION

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This brief commentary chapter devoted to issues suggested by the Mason and Lee chapters raises a number of fundamental questions concerning generalization: the epistemological status of generalization and the nature and complexities of generalization as it is manifested in the didactic context of the algebra classroom.

1. THE EPISTEMOLOGICAL STATUS OF GENERALIZATION

Mrs. Smith, sitting in her living room, hears the doorbell ring. She gets up to see who is at the door. No one is there, and she returns to the room. Mr. Smith pursues their conversation when once again the doorbell rings. Mrs. Smith gets up once again to answer, but no one is there. This scene repeats itself a third time. The fourth time the doorbell rings, Mrs. Smith exclaims to her husband: "Do not send me to open the door! You have seen that it is useless! Experience has shown us that when we hear the doorbell, it implies that no one is there!"

You surely remember the above scene from *Cantatrice chauve* by Eugne Ionesco. What is so captivating about this scene is that it reflects in an impeccable way the dynamics of a procedure of generalization (in fact, Mrs. Smith remains faithful to the "observed facts") and surely the conclusion is absurd (for us).

The fragile status of knowledge obtained through a generalization process brings us to the question of what constitutes a "good" or "bad" generalization. The answer to this question, which has puzzled mathematicians and philosophers for the last 25 centuries, has been given to us in various forms: normative logic, inductive or probabilistic logic, statistics, and so on. Surely generalization is not specific to mathematics: From a certain point of view, it is perhaps one of the deepest characteristics of the whole of scientific knowledge and even, perhaps, of daily non-scientific knowledge, as shown by the little extract of Ionesco's theater piece quoted above.

From a mathematical teaching perspective that favors generalizing activities, it may be convenient to try to answer the question: Why, in the construction of his/her knowledge, does the cognizer make generalizations? The "why" should be understood, of course, in its deeper meaning, so that we may specify the epistemological role of the generalization as well as the nature of the relation between generalization and the resulting knowledge.

From the same perspective, other interesting questions are:

- What is the significance of generalization in mathematics? and more specifically:
- What are the kinds and the characteristics of generalizations involved in algebra?
- What are the algebraic concepts that we can reach through numeric generalizations?

These questions are not addressed directly by Mason and by Lee (this volume). The reason is, it seems to me, that they give to generalization a particular epistemological status. Mason, for instance, says that "generalization is the life-blood, the heart of mathematics." Lee states that the most important activities related to algebra can be seen as generalizing activities; "nor is it much of a challenge," she says, "to demonstrate that functions, modeling and problem solving are all types of generalizing activities, that algebra and indeed all of mathematics is about generalizing patterns." The basic (more or less implicit) argument of both authors is that this "inner" developmental characteristic of mathematical activity, shown in particular by the history of mathematics (see references to history by both authors), can be translated in the field of education and used as a didactic device when mathematics is seen as a subject to be taught. I think that the hypothesis that generalization can be seen as an *epistemic norm* needs to be studied in greater detail and that the consequences that it has for the teaching of mathematics need to be specified. I believe that the answers to the above questions depend on the way in which we interpret the development of mathematics and the way in which we conceive the development of mathematical knowledge. A superficial look at the history of mathematics leads us to the impression that all mathematics is about generalizing. A closer look suggests that, if we accept generalization as an epistemic norm, it could not function alone but may be related to another probable epistemic norm, namely the problem-solving epistemic norm. Put roughly, I think that the latter functions as a primary need for knowledge, while the former functions as a driven-norm. Certainly, this is a point in mathematical cognition that requires deeper study.

2. WHAT ARE THE KINDS AND THE CHARACTERISTICS OF GENERALIZATIONS INVOLVED IN ALGEBRA?

In considering generalization from a didactic point of view, we should take into account that generalization depends on the mathematical objects we are generalizing. Generalization is not a context-free activity. There are many kinds of generalizations that can all be very different.

What are the characteristics of generalization based on geometric-numerical patterns? I would like to point out two specific elements in these kinds of generalizations. The first deals with a logical aspect and the second is related to the role played by external representations in generalizing geometric-numeric patterns.

2.1. *The Problem of Validity in Generalizing Results*

A goal in generalizing geometric-numeric patterns is to obtain a new result. Conceived in this form, generalization is not a concept. It is a procedure allowing for the generation, within a theory and beginning with certain results, of new results.

A generalization procedure g arrives at a conclusion α, starting from a sequence of "observed facts," $a_1, a_2, \ldots, a_n$. We can write this as:

$$a_1, a_2, \ldots, a_n \dashrightarrow \alpha \qquad (\alpha \text{ is derived from } a_1, a_2, \ldots, a_n)$$

The facts a_1, a_2, ..., a_n are interpreted according to a certain way of thinking (Reck, 1981, refers to a style of thinking), depending on the knowledge and purposes of the observer.[1] This way of thinking results from the observer's conceptualization of the mathematical objects and relations involved in and between the facts a_i, and leads to a particular form of mathematical thinking.

What is important here is that one of the most significant characteristics of generalization is its logical nature, which makes possible the conclusion α. The underlying logic of generalization can be very different, depending on the student's mathematical thinking. For instance, many students think that some examples (even one or two) are sufficient to justify the conclusion α. (Radford & Berges, 1988). Other students think that guessing the result from the first terms a_1. a_2. a_3 of a sequence is sufficient to justify the conclusion α. Other students think that the validity of a conclusion α is accomplished by testing it with a special term of the sequence, let's say the 100th term, or even the 1000th term. After all, is it necessary to prove a., when it appears as an obvious statement?[2] Who decides about validity?

Generalization as a didactic device cannot avoid the problem of validity, and validity is in itself a very complex idea. This does not mean that generalization cannot be a useful bridge to algebra. I want to point out that using generalization supposes that we should be prepared to work with this additional (logical) element in the classroom.

2.2. *External Representations as Symbols*

The use of external representations as symbols in generalization will be different from their use in elementary arithmetic. This is, in my opinion, another aspect to take into consideration. In order to explain this, let e_1, e_2, ..., e_n be the symbolic expressions of the facts a_1, a_2, ..., a_n. The e_is are "sentences" in a symbolic system L_1 (take as an example a certain arithmetic symbolic system, e.g., the modem arithmetic symbolic system or the ancient Greek arithmetic symbolic system) and let ε be the symbolic expression of a. in a certain symbolic system L_2 (eventually the same symbolic system L_1 or another one, e.g., our modem algebraic symbolic system), then a generalization procedure can be seen as illustrated in Figure 1.

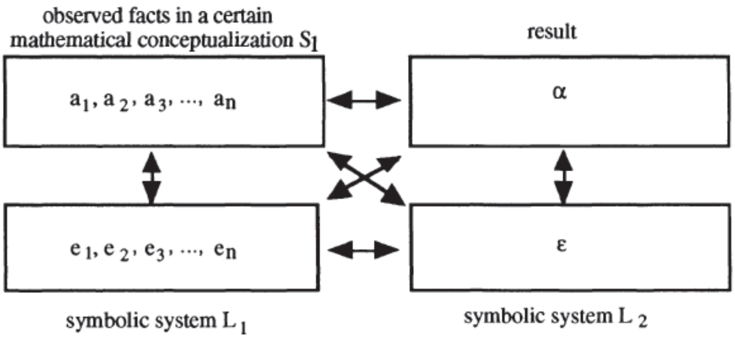

Figure 1.

The diagram shown in Figure 1 displays the many functioning modes of a generalization procedure. In the *How Many?* activity of Mason's appendix (this volume), he places emphasis on the double relation S_1 <--> L_1, in order to obtain the sequence of e_is: 1, 1 + 3 x 1, 1 + 3 x 2, 1 + 3 x 3. Of course, in doing so, one of the most important difficulties a student must face is to understand this very special way of counting, in which we do not write the actual number of rectangles (i.e., 1, 4, 7, 10, ...), but new expressions of these numbers. It requires "seeing" the "facts" a_i in a different way. Representations as mathematical symbols are not independent of the goal. They require a certain anticipation of the goal. The problem that now arises is that of knowing which facets of the object should be kept in its representation.[3]

We will now try to see, in "slow motion," the mental jump to larger terms in the sequence from this activity.

When the question is asked: how many squares will be used to make the 137th picture, the generalization is done from the invariance of the syntactic structure of the e_is (the student can then get the answer 1 + 3 x 136). But how can she be sure that this invariance is an acceptable argument?

We can take advantage of the above diagram to see an important difference between the questions asked by Lee and those found in Mason's appendix. When referring to the "dot rectangle problem," the first two questions are asked within arithmetic. The last question cannot be asked within arithmetic. In fact, in arithmetic we cannot make reference to the *n*th rectangle! This is a new expression related to a new concept. This last question is thus formulated within a domain where symbolic vehicles give precision to algebraic ideas. Even if both authors agree on the fact that algebraic symbolism is not the first goal of generalization, it is clear that this symbolism will be called upon to play a certain role.

For example, if in a certain pattern we see the sequence of terms in a certain way, we get a resulting expression $\varepsilon(n)$ of a formula α. But if we see this same sequence in another way, we get a different expression $\varepsilon'(n)$ for the same formula α. Take, for instance, the problem in Mason's appendix, where we get the formulas $\varepsilon(n) = 1 + 3(n-1)$ and $\varepsilon'(n) = 3n - 2$. What arguments can we use to show the equivalence of these formulas? Will these be syntactic arguments?

3. WHAT ARE THE ALGEBRAIC CONCEYTS THAT WE CAN REACH THROUGH NUMERIC GENERALIZATIONS?

Let's now consider the kind of concepts that can be reached in generalizing geometrical-numeric patterns. The goal of these generalizations is to find an expression ε representing the conclusion α. The expression ε is in fact a formula and is constructed on the basis, not of the concrete numbers (like 1, 4, 7) involved in the first facts observed, but on the idea of a general number. General numbers appear as preconcepts to the concept of variable.

On the other hand, the goal in algebraic problem solving (where "problem" designates a word problem) is not to find a formula, but a number (i.e., unknown) through an equation. We therefore have a different situation than the case of generalizing a pattern. But this difference is not uniquely at the word level, which would lead us to believe that an unknown is but a variable, and an equation is only a type of formula. The difference is in fact a fundamental difference--a conceptual difference--that often goes unnoticed (school books and even school guidelines frequently do not recognize this difference).

In fact, the logical base underlying generalization is that of justifying the conclusion. It is a proof-process, which moves from empirical knowledge (related to the facts a_i) to ·abstract knowledge that is beyond the empirical scope. Yet, the logical base of algebraic resolution is found in its analytic nature. This signifies that when solving an equation, or a word problem, we are supposing that we know the number we are looking for, and we handle this number as if it were known, so that we can reveal its identity in the end. Therefore, we must place ourselves in a hypothetical situation. We therefore realize that the logical bases are, in both cases, very different. The generalization way of thinking and the analytic way of thinking that characterizes algebraic word problem solving are independent and essentially irreducible, structured forms of algebraic thinking.[4]

The above discussion suggests that the algebraic concepts of *unknowns* and *equations* appear to be intrinsically bound to the problem-solving approach, and that the concepts of *variable* and *formula* appear to be intrinsically bound to the pattern generalization approach.[5] Thus, generalization and problem-solving approaches appear to be mutual complementary fields in teaching algebra. How can we connect these approaches in the classroom? I think this is an open question.

NOTES

Lee's chapter shows, in the "dot rectangle problem," several kinds of perceptions or interpretations of the facts a_i (a_i being the sequence of dot rectangles).

2 Take, for instance, Lee's first problem, where students had to show, using algebra, that the sum of two consecutive numbers is always an odd number.

3 The results obtained by Lee in the consecutive numbers problem are quite eloquent in this respect: "+1" is difficult to perceive as an even number, since the symbol "+1" suggests an excess that is not compatible with the idea of even numbers.

4 It does not mean that a generalization task cannot lead to the solving of an equation or vice versa. For instance, in the formula E of Mason's problem $N = 1+3\,(n - 1)$, we can ask the following question:

What is the rank of the figure with 598 squares? What we claim here is that when the student engages herself in an algebraic procedure in trying to solve the equation $598 = 1 + 3\,(n - 1)$, the intellectual process will be supported by a different logical basis using concepts belonging to a different "form" of thinking than that used in the process of obtaining the formula ε.

5 There is another element that points to this same conclusion. If we look at the emergence of symbolism from a historical perspective, we notice that the use of the unknown in problem solving has often led to the development of different algebraic languages (Diophantus, Chuquet, Viete, etc.). However, the symbolic representations for the concept of variable came much later: Historically, the mathematical objects of variable and equation come from different conceptualizations (see *The Roles of Geometry and Arithmetic in the Development of Algebra: Historical Remarks from a Didactic Perspective* in this volume).