EMBODIMENT, PERCEPTION AND SYMBOLS IN THE DEVELOPMENT OF EARLY ALGEBRAIC THINKING

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Placed in the context of early algebra research, this article deals with the question of the development of algebraic thinking in young students. In contrast to mental approaches to cognition, we argue that thinking is made up of material and ideational components—such as (inner and outer) speech, forms of sensuous imagination, gestures, tactility, and actual actions with signs and cultural artifacts. Drawing on data from a longitudinal classroom based research program where 8-year old students were followed as they moved from Grade 2 to Grade 3 to Grade 4, our developmental research question is investigated in terms of the manner in which new relationships between embodiment, perception, and symbol-use emerge and evolve as students engage in patterning activities.

INTRODUCTION AND THEORETICAL FRAMEWORK

The idea that young students—even with limited knowledge of arithmetic— can start learning some algebraic concepts has received increasing experimental support in the past few years. It has been found that, with suitable instructional support, young students can understand some aspects of pattern generalization—e.g., to describe the terms of a sequence according to the position they occupy therein (e.g., Becker & Rivera, 2008; Moss & Beatty, 2006; Warren & Cooper, 2008). However, despite the growing research on early algebra, many research questions remain open. For instance, little is known about how algebraic thinking develops in young students. This article seeks to contribute to this research question through the analysis of data collected in the course of a 3-year longitudinal classroom based research program.

Generally speaking, developmental questions of the kind we are dealing with here are not easy to investigate. They can only be formulated and tackled against the backdrop of explicit theoretical views about thinking and development. In our case, our research is framed by a theoretical Vygotskian perspective on teaching and learning—the *theory of knowledge objectification* (Radford, 2008a). A central feature of this theory is that, in contrast to mental cognitive approaches, thinking is not considered as something that solely happens ‘in the head.’ Thinking is rather considered as made up of material and ideational components: it is made up of (inner and outer) speech, objectified forms of sensuous imagination, gestures, tactility, and our actual actions with cultural artifacts. This does not mean that thinking is a *collection* of items. We consider thinking as a dynamic *unity of material and ideal components*—a tangible social practice materialized in the body (e.g. through kinaesthetic actions, gestures, perception, visualization), in the use of signs (e.g. mathematical symbols, graphs, written and spoken words), and artifacts of different

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sorts (rulers, calculators and so on). Within this context, to ask the question of the development of algebraic thinking is to ask about the appearance of new structuring *relationships* between the material-ideational components of thinking (e.g., gesture, inner and outer speech) and the manner in which these relationships are organized and reorganized. Now, in the theoretical perspective articulated here, development is not considered to follow any pre-established or innate path. Rather, it is considered to be cultural through and through. Our research question therefore does not simply concern the appearance of new forms of psychic functioning but also the contextual conditions that make these forms possible in the first place. It is against this theoretical framework that the question of the development of young students' algebraic thinking is investigated in the following sections.

METHODOLOGY: DATA COLLECTION AND ANALYSIS

Our data comes from a 3-year longitudinal research program conducted in an urban primary school in which a class of 25 8-year old students was followed as the students moved from Grade 2, to Grade 3, to Grade 4. The data was collected during regular mathematics lessons designed by the teacher and our research team. To collect data, we used four or five video cameras, each filming one small group of students (groups of 2 or 3). The data that is presented here comes from episodes of what happened when the students were dealing with questions about pattern generalization. We focus in particular on a student, Carlos, whose developmental path is representative of our findings. In tune with our theoretical framework, to investigate the development of early algebraic thinking we conducted a *multi-semiotic data analysis*. Once the videotapes were fully transcribed, we identified salient episodes of the activities. Focusing on the selected episodes, we carried out a low-motion and a frame-by-frame fine-grained video microanalysis to study the role of and the relationship between gestures, language, and mathematical signs.

RESULTS AND DISCUSSIONS

First episode: Grade 2

The first algebra activity that the students tackled in Grade 2 revolved around the sequence shown in Fig. A.

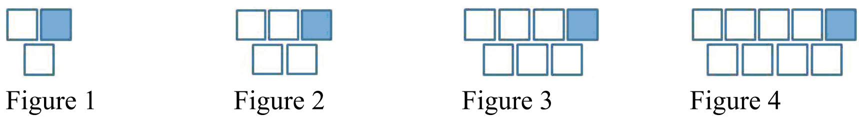

Fig. A. The first four figures of a sequence given to the students in a Grade 2 class.

In the first part of the activity, the students were asked to extend the sequence up to Figure 6. Carlos, one of the students, started counting the squares aloud, accompanying the counting process with a rhythmic upper body movement and pen-pointing gestures. He counted all the squares in an orderly way, beginning with the squares in the top row, from left to right, then those in the bottom row (see Fig. B, pic. 1-2). Then he drew Figure 5 in an orderly manner, starting from the bottom row, left to right. Although Figure 5 contains almost the right number of squares, it certainly does not conform to the two-row arrangement of the given terms of the sequence (See Fig. B, pic. 3).

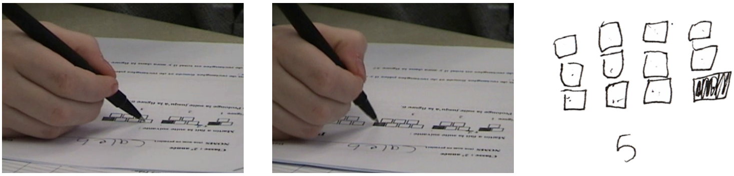

Fig. B. In pics 1 and 2, while counting aloud, Carlos sequentially points to the squares in the top row of Figure 3. Pic 3 shows Carlos's Figure 5.

To come up with an interpretation of Carlos's actions, let us note that, generally speaking, to extend a figural sequence, the students need to grasp a regularity that involves the linkage of two different structures: one *spatial* and the other *numerical*. From the spatial structure emerges a sense of the squares' *spatial position*, whereas their numerosity emerges from a numerical structure. While Carlos attends to the numerical structure in the generalizing activity, the spatial structure is not coherently emphasized. This does not mean that Carlos does not see the figures as composed of two horizontal rows. As in the case of other students, Carlos's emphasis on the numerical structure somehow leaves in the background the geometric structure. This emphasis reappeared when he finished drawing Figure 5: since the shape of the figure did not provide him with a clue about its numerosity, he might have felt the need to count the squares again. We could say that the *shape* of the terms of the sequence is used to facilitate the counting process (as he always counted the squares in a figure in an spatial orderly way), but that the geometric structure does not come to be related to the numerical one in a meaningful efficient way. Carlos's process can be contrasted to Kyle's, where shape is emphasized but numerosity is not well attended. Kyle drew Figure 5 as having two rows but drew 4 squares on the bottom and 4 squares on the top row. These examples—as well as those reported by Rivera (2010) with other Grade 2 students—suggest that the linkage of spatial and numerical structures constitutes an important aspect of the development of algebraic thinking. That such a linkage is less natural than it may appear at first sight can be made evident if we resort to studies in special education. It is well known that children with Down syndrome tend to reproduce figures such as Figure 5 in terms of their shape without much attention to numerical details; in contrast, children with Williams syndrome tend to present more analytical thinking, which focuses on the numerical in detriment to the spatial (Brigaglia, 2010). Or as Bellugi, Lai, and Wand (1997) note, William Syndrome subjects are typically impaired at reproducing global forms, while Down Syndrome subjects tend to produce global forms without local information. Coming back to our Grade 2 students, it is interesting to note that in extending the sequences, the students did not use deictic spatial terms, like "bottom" or "top." (There was one exception: Kyle, who talked once about the "top row," without hence using it in a systematic manner.) In the cases in which the students did succeed in linking the spatial and numerical structures, the spatial structure appeared ostensibly only, i.e., in the embodied realm of action and perception (Radford, 2010, in press). The geometric structure reached the realm of language the next day, when the teacher discussed the sequence with the students. Indeed, during the debriefing of the first day, it was agreed with the teacher that it would be important to bring to the students' attention the linkage of the numerical and spatial structures. To do so, the teacher drew the first five terms of the sequence on the blackboard and refereed to an imaginary student who counted by rows: "This student," she said to the class, "noticed that in Figure 1 (she pointed to the name of the figure) there is one rectangle on the bottom (and she pointed to the rectangle on the bottom), one on the top (pointing to the rectangle), plus one dark rectangle (pointing to the dark rectangle)." Next, she moved to Figure 2 and repeated in a rhythmic manner the same counting process coordinating the spatial deictics "bottom" and "top," the corresponding spatial rows of the figure, and the number of rectangles therein. To make sure that everyone was following, she started again from Figure 1 and, at Figure 3, she invited the students to join her in the counting process, going together up to Figure 5 (see Fig. C).

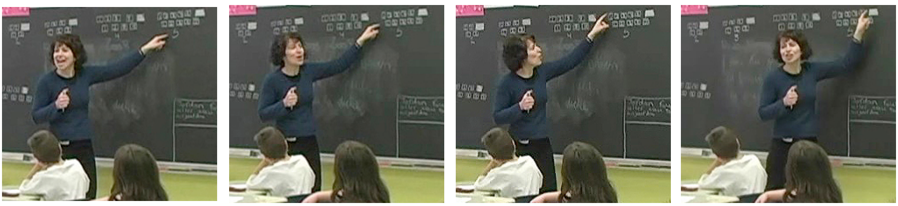

Fig. C. The teacher and the students counting rhythmically say (see Pic 1) "Figure 5", (Pic 2) "5 on the bottom", (Pic 3) "5 on top", (Pic 4) "plus 1."

Then, the teacher asked the class about the number of squares in Figure 25. Mary raised her hand and answered: "25 on the bottom, 25 on top, plus 1." The class spent some time dealing with "remote" figures, such as Figure 50, and 100. Schematically speaking, the students' answers were "x on the bottom, x on the top, plus 1" where x was always a *specific* number. Since at the time the students were able to make systematic additions up to 25, the teacher made calculators available to them and asked the students to explain the steps to calculate the total of squares in specific figures. Schematically speaking, the students' answer was "x + x +1" (where x was always a *specific* number). The students came back to small-group work and continued their work. In one of the questions, they had to explain how Pierre should proceed to build a big figure of the sequence. The goal of this and other similar questions was to give an opportunity to the students to objectify a numerical-spatial regularity of the given terms of the sequence and to use it to imagine and deal with remote (or even unspecified) terms. Carlos wrote: "Pierre wants to build Figure 10,000. Pierre has to put 10,000 on the bottom[;] on the top he has to put 10,001." In our PME 34 paper we dealt with the nature of the students' emergent algebraic thinking. What we want to discus here is the question of development. As stated in our theoretical framework, conceptual development is marked by the appearance of new *relationships* between the material-ideational components of thinking; it brings forward new forms of psychic functioning. If during the first day Carlos and other students were emphasizing the analytic process of counting squares one by one, from the second day on, their perception of the figures and the counting processes changed. The link between the spatial and geometric structures was achieved and, as illustrated by Mary's and Carlos's answers, spatial deictics became part of their linguistic repertoire. These changes bear witness to the appearance of new relationships between gesture, speech, perception, imagination, and counting. A new unity of material and ideational components of thinking was forged. Thus, the students were able not only to imagine remote figures (e.g., Figure 100)—which would be difficult to imagine within the relationships of ideal and material components of thinking underpinning pure analytic, one-by-one counting procedures— but also to devise formulas to calculate the number of squares in figures beyond perception (e.g., "100+100+1").

The joint counting process in which the teacher and the students engaged during the second day is, of course, an instance of a *zone of proximal development*. The explicit use of rhythm, gestures, and linguistic deictics by the teacher, followed later by the students, opened up new possibilities for the student to use efficient and evolved cultural forms of mathematical generalization that they successfully applied to other sequences with different shapes. The joint counting process made it possible for the students to *notice* and *articulate* new forms of mathematical generalization. In particular, they became aware of the fact that the counting process can be based on a relational idea: to link the number of the figure to relevant parts of it (e.g. the squares on the bottom row). This requires an altogether new perception of the number of the figure and the figures themselves. The figure appears now not as a mere bunch of ordered squares but as something susceptible of being decomposed, the decomposed parts bearing potential clues for algebraic relationships to occur. But it is not only perception that is developmentally modified. In the same way as perception develops, so do speech (e.g., through spatial deictics) and gesture (through rhythm and precision). Indeed, perception, speech, gesture, and imagination develop in an interrelated manner. They come to form a new unity of the material-ideational components of thinking, where words, gestures, and signs more generally, are used as means of objectification, or as Vygotsky put it, "as means of voluntary directing attention, as means of abstracting and isolating features, and as a means of […] synthesizing and symbolising" (1987, p. 164).

Second episode: Grade 3

As usual, in Grade 3 the students were presented with generalizing tasks to be tackled in small groups. The first task featured a figural sequence, *Sn*, having *n* circles horizontally and *n-1* vertically, of which the first four terms were given. Contrary to what he did first in Grade 2, since the outset, Carlos perceived the sequence taking advantage of the spatial configuration of its terms. Talking to his teammates about Figure 4 he said: "here (pointing to the vertical part) there are four. Like you take all this [i.e., the vertical part] together (he draws a line around), and you take all this [i.e., the horizontal part] together (he draws a line around; see Fig. D, pic 1). So, we should draw 5 like that (through a vertical gesture he indicates the place where the vertical part should be drawn) and (making a horizontal gesture) 5 like that" (see Fig. D, pics 2-3).

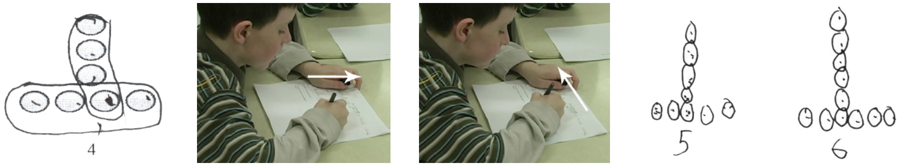

Fig. D. To the left, Figure 4 of the given sequence. Middle, Carlos's vertical and horizontal gestures while imagining and talking about the still to be drawn Figure 5. To the right, Carlos's drawings of Figures 5 and 6.

When the teacher came to see the group, she asked Carlos to sketch for her Figure 10, then Figure 50. The first answer was given using unspecified deictics and gestures. He quickly said: "10 like this (vertical gesture) and 10 like that" (horizontal gesture). The specific deictic term "vertical" was used in answering the question about Figure 50. He said: "50 on the vertical… and 49…" When the teacher left, the students kept discussing how to write the answer to the question about Figure 6. Carlos wrote: "6 vertical and 5 horizontal." In developmental terms, we see the evolution of the unity of ideational-material components of algebraic thinking. Now, Carlos by himself and with great ease coordinates gestures, perception, and speech. The coordination of these outer components of thinking is much more refined compared to what we observed in Grade 2. This refinement is what we have called a *semiotic contraction* (Radford, 2008b) and is a symptom of learning and conceptual development.

Third episode: Grade 4

To check developmental questions, in Grade 4 we gave to the students the sequence with which they started in Grade 2 (see Fig. A). This time, from the outset, Carlos perceived the terms as being divided into two rows. Talking to his teammates and referring to the top row of Figure 5, he said as if talking about something banal: "5 white squares, 'cause in Figure 1, there is 1 white square (making a quick pointing gesture) … Figure 2, 2 [squares] (making another quick pointing gesture); 3, (another quick pointing gesture) 3." He drew the five white squares on the top row of Figure 5 and added: "after that you add a dark square." Then, referring to the bottom row of Figure 4: "there are 4; there [Figure 5] there are 5." When the teacher came to see their work, Carlos and his teammates explained "We looked at Figure 2, it's the same thing [i.e., 2 white squares on top] . . . Figure 6 will have 6 white squares."

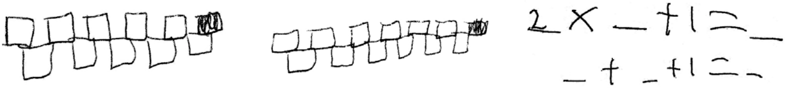

Fig. E. Left, Carlos' drawings of Figures 5 and 6. Right, Carlos's formulas.

In his answer to the question about explaining what Pierre has to do to build a big figure of the sequence, Carlos wrote: "He needs [to put as many white squares as] the number of the figure on top and on the bottom, plus a dark square on top." The algebraic formula that he provided is shown in pic 3 of Fig. E. From a developmental perspective, we see how Carlos's use of language has been refined. In Grade 2 he was resorting to particular figures (Figure 1,000) to answer the same question. Here he deals with indeterminacy in an easy way, through the expression "the number of the figure." He even goes further and produces two symbolic expressions to calculate the total of squares in the unspecified figure.

SYNTHESIS AND CONCLUDING REMARKS

This paper seeks to contribute to the question of the development of young students' algebraic thinking. Framed by the theory of objectification, it was suggested that thinking is a *unity of material and ideal components*—inner and outer speech, forms of sensuous visualization and imagination, gestures and tactility, etc. Development is considered to consist of the refinement of previous, and the appearance of new, structuring *relationships* between the material-ideational components of thinking. Within this framework, early algebraic thinking is considered to be based on the student's possibilities to grasp patterns in culturally evolved co-variational ways and use them to deal with questions of remote and unspecified terms. Cognitively speaking, for this to occur, the students have to resort to a coordination of numeric and spatial structures. The awareness of these structures and their coordination entail a complex relationship between (inner or outer) speech, forms of visualization and imagination, gesture, and activity on signs (e.g., numbers and proto-algebraic notations). Our data offer a glimpse of the evolution of algebraic thinking. It shows how in Grade 2 "spontaneous" perception was successfully transformed through the joint work of the teacher and the students. This joint work, we suggested, might be conceptualized as occurring in a zone of proximal development out of which the students created new psychological functions. For as Schneuwly notes, "Teaching does not implant new psychological functions in the child. It makes the tools available and creates the conditions necessary for the child to build them" (1994, p. 288). A substantial refinement of the new *relationships* between the material-ideational components of algebraic thinking was accomplished in Grade 3. In Grade 4 we witnessed the marked evolution of language to deal with indeterminate terms of sequences and the appearance of a new component—symbolic activity (see Fig. E). At this point we are investigating how the new symbolic activity gives rise to abstract algebraic notations.

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References

Becker, J., & Rivera, F. (Eds.) (2008). Generalization in algebra: the foundation of algebraic thinking and reasoning across the grades. *ZDM, 40*(1).

Bellugi, U., Lai, Z., & Wang, P. (1997). Language, communication. and neural system in Williams syndrome. *Mental Retardation and Developmental disabilities, 3*, 334-342.

Brigaglia, P. (2010). *Insegnamento/apprendimento della geometria in situazioni di disabilità.*. Università di Palermo, Unpublished Ph.D. Dissertation. Palermo, Italy.

Moss, J., & Beatty, R. (2006). Knowledge building and knowledge forum: Grade 4 students collaborate to solve linear generalizing problems. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proc. 30th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 193-199). Prague: PME.

Radford, L. (2008a). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring & F. Seeger (Eds.), *Semiotics in mathematics education.* (pp. 215-234). Rotterdam: Sense Publishers.

Radford, L. (2008b). Iconicity and contraction: A semiotic investigation of forms of algebraic generalizations of patterns in different contexts. *ZDM, 40*(1), 83-96.

Radford, L. (2010). Elementary forms of algebraic thinking in young students. In M. F. Pinto & T. F. Kawasaki (Eds.), *Proc. 34th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 73-80). Belo Horizonte: PME.

Radford, L. (in press). Grade 2 students' non-symbolic algebraic thinking. In J. Cai & E. Knuth (Eds.), *Early Algebraization*. Berlin: Springer-Verlag.

Rivera, F. D. (2010). Second grade students' preinstructional competence in patterning activity. In M. F. Pinto & T. F. Kawasaki (Eds.), *Proc. 34th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 81-88). Belo Horizonte: PME.

Schneuwly, B. (1994). Contradiction and development: Vygotsky and paedology. *European Journal of Psychology of Education, 9*(4), 281-291.

Vygotsky, L. S. (1987). *Collected works (Vol. 1)*. R. W. Rieber, & A. S. Carton (Eds.). New York: Plenum.

Warren, E., & Cooper, T. (2008). Generalising the pattern rule for visual growth patterns. *Educational Studies in Mathematics, 67*, 171-185.