ARITHMETICAL AND ALGEBRAIC THINKING IN PROBLEM-SOLVING[1]

Nadine Bednarz, Luis Radford, Bernadette Janvier, André Lepage

CIRADE, Université du Québec à Montréal

The fact that students have difficulty acquiring and developing algebraic procedures in problem-solving, considering the arithmetical experience that they have acquired over years, calls for a didactic reflection on the nature of the conceptual changes which mark the transition from one mode of treatment to the other. In this perspective, our study seeks to characterize the spontaneous problem-solving strategies used by Secondary III level students (14 -and -15-year-olds), who have already taken one algebra course, when solving different problems. The analysis of the problem-solving procedures developed by these students reveals the differences between the conceptual basis which underlie the two modes of thought.

The difficulties experienced by students learning algebra have been the subject of many studies which have shown that certain conceptual changes are necessary to make the transition from arithmetic to algebra (Booth, 1984; Collis, 1974, Kieran, 1981; Filloy and Rojano, 1984; Hercovicz and Linchevski, 1991; Arzarello, 1991...). In the area of problem-solving, which is one of the important heuristic functions of algebra (Kieran, 1989) and which proves very difficult for students (Lochead, 1988; Kaput, 1983; Clément, 1982; Mayer 1982), the analyses examining the passage to an algebraic mode of thinking have either focused on a certain dialectic between procedural and relational thought (Kieran, 1991; Arzarello, 1991), or on the symbolism and/or the solving of equations. With regard to the latter, the history of mathematics shows that algebra began to develop well before symbols were used to represent unknown quantities. Rhetoric was an important stage in this development among the Arabs, for whom language was the natural means to represent the (known and unknown) quantities of a problem to be solved and to express the solution process. The studies carried out among students also show that most of them, from high school to university, solve algebraic problems in an "abridged" style (natural syncopated language) rather than in a symbolic style (Kieran, 1989; Harper, 1979). Few studies, however, have focused on the students' *reasoning* in solving the problems.

---

[1] This study is part of a larger project undertaken by a group from CIRADE, subsidized by the FCAR (Quebec), which is researching the conditions for the construction of algebraic reasoning and representations, with regard to the situations which allow for their emergence and development.

In our didactic perspective, the main objective of our analysis was to gain a better understanding of the *conceptual basis* which underlie the arithmetical mode of thought on one hand, and the algebraic mode of thought on the other, as well as the possible articulation-conflicts which are possible in the transition from one mode of treatment to the other.

From a *didactic point of view,* because of the previous experience acquired by the students, problem-solving appears to be an interesting terrain for examining the two modes of thought and the conceptual changes which mark the passage from the arithmetical to the algebraic thinking. Moreover, from the *historical point of view,* the solving of problems played an important role in the development of algebra. It is at the heart of the algebra of Diophantus and of the Arabs, and is explicit in Vieta's objective of developing a method that could solve every problem. Thus, problem-solving is a doubly interesting terrain for the examination of the emergence of the algebraic mode of thought and its characteristics. This historical analysis is now being carried out, and is the object of investigation of some of the members of our team (Charbonneau, 1992; Lefebvre, 1992; Radford, 1992).

Objective of the Study

By examining the ways in which secondary school students (Sec. III, 14-and-15-year-olds who had already taken an algebra course) spontaneously solved different types of problems, this exploratory research project, carried out with a small group of students, aimed to analyse the solution processes of the students. In the characterization of the arithmetical and algebraic procedures used, the accent was placed not on the use of symbolism, but rather on the students' capacity to grasp the known and unknown quantities in the problem, and their way of solving it.

Method

In order to delineate, on an exploratory basis, the procedures used by students, and, through these, to better elucidate the differences between the *conceptual basis* which underlie the arithmetical and the algebraic thinking, 54 students from two regular classes in a Montreal area public high school (Secondary III, 14-and-15-year-olds) were given a paper-and-pencil test with five different written problems to solve[2]. The choice of the students' level (they had taken an introductory course in algebra) made it

[2] The different problems presented, involving complex relations, could all be solved a priori by either arithmetical or algebraic reasoning, even if some of the methods appear more complicated than others.

possible for us to show the conflicts that can arise for students at this stage when facing two possible modes of solving the problems.

Analysis of the Results

Our analysis centered on one of the problems, which read as follows: "588 passengers must travel from one city to another. Two trains are available. One train consists only of 12-seat cars, and the other only of 16-seat cars. Supposing that the train with 16-seat cars will have eight cars more than the other train, how many cars must be attached to the locomotives of each train ?"

In this problem, different solution processes were possible. These took into account a certain implicit mental representation of the data and the relations which linked the elements involved, a representation which evolved during the solution process. How can we distinguish between the arithmetical and the algebraic procedures in the ways that this data and these relations were dealt with ?

A preliminary analysis of the above problem brought out the key elements around which the solution will be organized: "the total number of passengers: 588", the existence of "two trains", of "16-seat cars", "12-seat cars", and the "eight cars more" that one kind of train had in relation to the other.

However, the resolution of the problem required the use of other elements which made it possible to "build bridges" between the different data, elements which were not at all explicit in the problem: the number of 16-seat cars and 12-seat cars, the relation between the two types of quantities involved: the number of cars and the number of passengers, which must be built from the rates given in the problem, the number of passengers in each train... This a priori analysis brought to light important reference points which guided the subsequent analysis of the students' ways of solving the problem.

SOME ARITHMETICAL PROCEDURES

It was easily observed that certain elements were retained by the students, and that these were used as a kind of point of entry , or engagement, in the organization of their solution procedures: a) the **two** trains; b) the **whole**: the 588 passengers; c) the **difference** between the number of cars of one type and those of other type; d) the data: "16-seat cars" and "12-seat cars".

In general, the arithmetical procedures were organized around these four **known elements**, in attempts to build bridges between them to be able to work with known data. The unknown quantity therefore appeared at the end of the process. Two types of entry points, or engagements, were distinguished. In the first case, the first two elements (a and b) frequently gave rise to a numerical strategy which we call **equitable partition**, which consisted in dividing the number of passengers by the number of trains (in this case, two) to obtain the number of passengers in each train (see Procedure 3). Another less frequent type of engagement was the **adjustment of the difference** between the two trains (c) at the beginning, to obtain two trains having the same number of cars (see Procedures 1 and 2).

1. Procedure taking the difference into account at the beginning:

Student:	Comments:
8 x 16 = 128 passengers	Numbers of passengers in the 8 extra cars
588 - 128 = 460 passengers	By eliminating the extra cars, the number of cars in each train is equal
460 ÷ 28 = 16.4	12 seats + 16 seats = 28 seats (one 28-seat car train)

Answer:
17, 12-seat cars and 25, 16-seat cars

This strategy clearly showed the modifications which occurred in the representation of the problem during the solution process: this representation was not at all static. The problem, and the relations linking the data had to be transformed by the students into a new configuration of the whole, which made it possible for the calculations to progress. The arithmetical procedure used here, which only dealt with the known elements, could not advance without those necessary modifications, because at the beginning there was no relation directly linking the known quantities provided in the problem.

2. Another procedure taking the difference into account at the beginning, followed by partition:

Student:	Comments:
16 x 8 = 128	Numbers of passengers in the eight extra cars
588-128 = 460	Modification of the initial representation into a new configuration of equality of cars (see previous strategy)
460 ÷ 2 = 230	The equitable sharing strategy
(230 ÷ 12 = 19, 230 ÷ 16 = 14)	Number of cars of each type
12-seat car train -> 19 cars	
16-seat car train -> 14 + 8 = 22 cars	Re-utilization of the difference

Just as in the first procedure, the representation of the problem was modified during the solving process. The change from the initial representation of inequality to one of equality authorized the use of equitable partition schema.

3. Procedure with partition at the beginning:

Student:	Comments:
588 ÷ 2 = 294	Division by two of the given total
294 ÷ 16 = 19	Calculation of the number of 16-seat cars
19 + 8 = 27	The use of the difference
27 x 16 = 432	Number of passengers in the 16-seat car train
588 - 432 = 156	Calculation of the number of passengers in the 12-seat car train
156 ÷ 12 = 13	Number of 12-seat cars

The more frequent recourse to the equitable partition schema at the beginning suggested a less complicated representation than the preceding one, in which the inequality of the number of cars had to be taken into consideration. The students' errors in all of the arithmetical procedures occurred precisely in the coordination of the equitable partition schema and the inequality of the number of cars.

PROCEDURES BETWEEN ALGEBRA AND ARITHMETIC (revealing a process in formation)

In the following strategy (see Procedure 4), after undertaking an arithmetical method, the student subsequently abandoned it, and wrote an equation. The solution of the equation was used immediately afterward in a step which went back to an arithmetical procedure.

4. Student:	**Comments:**
Arithmetical trial: 16 x 8 =128	previous procedure which took the difference into account at the beginning
588 - 128 = 460	

Algebraic step: $x + 8x = 588$; $9x = 588$; $x = 65$

Arithmetical procedure:	
65 ÷ 2 = 32	equitable partition schema: the number of cars is divided by two
32 - 8 = 24	use of the difference
24 x 12 = 288	passengers travelling in the 12-seat cars

In this example, the student began by adjusting the number of passengers to arrive at two trains having an equal number of cars. In this arithmetical engagement, there is a semantic control of the situation and the relations which link the elements involved. When the student left this procedure in favour of an algebraic one, the continuation shows that there was no longer any control over the rates (which appear to be completely ignored) or the difference, although the algebraic treatment of the equation is correct. There was a complete loss of control over the situation. But as soon as the student returned to arithmetic, the control was regained. This and the following examples clearly show the distinctions effected by the student in the transition from one mode to the other. The passage to algebra requires the construction of a more global representation of the problem, which is in opposition to the sequence of dynamic representations which are the basis of the arithmetical reasoning.

5. Student:

Arithmetical trial: 16 x 8 =128; 588 - 128 = 460; 460 + 2 = 230 (Control of the situation)

Algebraic trial: 588 + 2 + 12 + 16 = 16 + 8x
578 = 16 + 8x
562 = 8x

Note that the order of the terms of the equation followed that of the presentation of the numbers in the text (loss of control - the student did not take into account the meaning of the quantities and of the problem).

6. Arithmetico-algebraic strategy:

Student:

8 x 16 = 128; 588 - 128 = 460 (difference taken into account)

Then the student switched to an algebraic mode, with the equation: 12x + 16x = 460, and ended by solving the problem.

ALGEBRAIC PROCEDURES

In contrast to the arithmetical procedures, in the algebraic procedures, the representation of the problem and the calculations do not generally undergo a parallel development. The solution process - which in arithmetic is based on a necessary transformation of the representation of the problem, in relation to meaning of the numbers obtained in successive calculations - needs at the beginning a representation of the relations between the data. It requires then for the student a global representation of the problem, from the start of the procedure, to infer an external symbolic representation modeling these relations, in the form here of an equation. Once the equation is expressed, the algebraic calculations often proceed independently of this representation of the situation. If the semantic control of the problem is re-established, it only happens at the end of the process. This type of engagement, totally different in its management of the data, is based on an element which is not present in arithmetic, that is, the introduction of precisely that quantity which is sought, the unknown quantity. We find there the analytical character of algebra so important to Vieta.

Student:

1st x 12
2nd (x+8)12 588 = x.12 + (x+8)16
588 = 12x + 16x + 128
-12x - 16x = 128 - 588
-28x = -460
28x = 460
x = 16.42
1st => 16.42 x 12 =197.04 2nd => (16.42 + 8) 16 = 390.72
1st 197.04 2nd 390.72

The equality -28x = -460, for example, cannot be interpreted in the context of the problem. This distance from the problem, necessary to proceed with the algebraic operations, makes it impossible, at this point, to verify if the results obtained concur with what is sought in the problem. A further effort must be expended to reinterpret the results from the symbolic operations.

The analysis of the students' errors in constructing their equations, throughout their procedures, showed that they did not take certain elements, such as rates, into account. Their symbolizations only retained certain aspects of what had to be represented.

CONCLUSION

This analysis brought out differences between the *conceptual basis* that underlie the arithmetical and the algebraic modes of thought.

Arithmetical reasoning is based on representations which are particular to it, and involves a particular relational process. The successive calculations which work with known quantities are effectively based upon the necessary transformation of the relations which link the elements present, requiring a constant semantic control of the quantities involved and of the situation.

In algebraic reasoning on the contrary, the relations expressed in the problem are integrated from the beginning into a global "static" representation of the problem, nevertheless requiring specific necessary representations for this. This engagement, which is quite different in its management of the data, is based on the introduction of the unknown quantity at the very beginning of the process, and requires a detachment from the meaning of both the quantities and the problem to solve it.

Our results suggest that the difficulty experienced in the transition from arithmetic to algebra occurs precisely in the construction of the representation of the problem.

References

Arzarello, F. (1991). Procedural and Relational Aspects of Algebraic Thinking. *Proceeding of PME XV*, Assisi, Italy, I, pp. 80-87.

Booth, L.R. (1984). *Algebra: Children's strategies and errors*. Windsor, U.K.: NFR-Nelson.

Charbonneau, L. (à paraître). Du raisonnement laissé à lui-même au raisonnement outillé: l'algèbre depuis Babylone jusqu'à Viète.*Bulletin de l'Association Mathématique du Québec*.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education*, 14, pp. 16-30.

Collis, K.F. (1974). *Cognitive development and mathematics learning*. Paper presented at the Psychology of Mathematics Workshop. Center for Science Education, Chelsea College, London.

Filloy, E., Rojano, T. (1984). From an arithmetical to an algebraic thought. *Proceedings of the sixth annual meeting of PME-NA*, Madison, pp. 51-56 .

Harper, E.W. (1979). *The child's interpretation of a numerical variable*, University of Bath, Ph. D. Thesis, 400 pages.

Hercovicz, N., Linchevski, L. (1991). Pre-algebraic thinking: range of equations and informal solution processes used by seventh graders prior to any instruction.*Proceedings of PME XV*, Assisi, Italy, II, pp. 173-180.

Kaput, J. Sims-Knight, J. (1983). Errors in translations to algebraic equations: Roots and implications. *Focus on Learning Problems in Mathematics*, 5, pp. 63-78.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, 12, pp. 317-326.

Kieran, C. (1989). A perspective on algebraic thinking. *Actes de la 13e conférence internationale «Psychology of Mathematics Education»*, 2, pp. 163-171.

Kieran, C. (1991). A procedural-structural perspective on algebra research. Proceedings of PME XV. Assisi, Italy, 2, pp. 245-253.

Lefebvre, J. (à paraître). Qu'est l'algèbre devenue? De Viète (1591) à aujourd'hui (1991), quelques changements clefs. *Bulletin de l'Association Mathématique du Québec*.

Lochead, J., Mestre, J. (1988). From words to algebra: Mending misconceptions. *The Ideas of Algebra, K-12*, NCTM Yearbook.

Mayer, R.E. (1982). Memory for algebra story problems. Journal of Educational Psychology, 74(2), pp. 199-216.

Radford, L. (à paraître). Diophante et l'algèbre pré-symbolique. *Bulletin de l'Association Mathématique du Québec*.