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BEFORE THE OTHER UNKNOWNS WERE INVENTED: DIDACTIC INQUIRIES ON THE METHODS AND PROBLEMS OF MEDIAEVAL ITALIAN ALGEBRA[1]

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À la mémoire de mon ami André Lepage.

§ 1 Introduction

One of the emerging approaches in contemporary mathematical education studies is one which concerns the historical construction of mathematical knowledge (cf. v. gr. Glaeser, 1981; Filloy et Rojano, 1984).

This approach, that of historical epistemological inquiry, helps us:

(a) to better understand the cognitive difficulties experienced by our students, as well as to better interpret the errors and incorrect conceptualizations that arise when they learn specific mathematical contents (Vergnaud, 1990, p. 16);

(b) to make more enlightened decisions concerning the knowledge being taught; in particular, it may give rise to new means of organizing and articulating this knowledge in the classroom[2].

Furthermore, the results of the didactic-historical epistemological inquiry can also lead to new paths of didactic research and provide us a deep understanding of the current concepts included in modern curricula.

Concerning this last point, it can be worthwhile to emphasize the role that historical research can play in service and pre-service teachers training programs. In fact, most of the time, the teachers' ideas about the mathematical content they teach derive from the only *contemporary mathematical formulation* of the content under consideration[3]. Now, the *contemporary formulation* is the result of a long process of conceptual changes and transformations and not necessarily the best starting point for students. However, lacking other alternatives, the *contemporary formulation* becomes a straightjacket in the choice of content to teach, in its organization, and in its articulation with other knowledges[4].

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[2] For an example of a teaching sequence for the equation of 2nd degree see Radford, forthcoming$_1$.

[3] For a general overview of this problem, see Chevallard, 1985.

[4] E. Barbin (1992, p. 576) provides a perfect example of the phenomenon to which we are referring: she comments on the difficulty that teachers experience when trying to teach the concept of the limit of a sequence. In the old mathematics programs the concept of limit was taught using the formal definition. The new French mathematics programs encourage

Where algebra is concerned, *contemporary formulation* favours, in particular, the 'symbolism' of algebra (Lefebvre, 1991/92); in this context, algebra is often seen as the mastering of a certain symbolic language so, right from the beginning, all efforts in the classroom are made for students to become competent in this language. Historically, however, the 'symbolism' (in its modern meaning, the one that we find in today's school texts) did not become the driving force of algebraic development until the Renaissance (that is, more than 30 centuries after the first algebraic ideas had seen the light of day!)

Is it possible to introduce algebra in school without having the *immediate* objective of mastering the modern symbolic language? When we ask student teachers this question, and ask them to elaborate a teaching sequence for the introduction of algebra excluding the use of the usual symbols (x, y, z, ...), they are dumbfounded; for them, algebra without symbols simply does not exist. Even though it is not a matter of students in Junior high school, in the construction of their knowledge, following the same path as the ancient mathematicians, it seems to us that the back-tracking, or the intellectual *dépaysement,* allowed by historical analysis furnishes teachers with some new reference points and a greater flexibility in their classroom choices.

However, resorting to the history of mathematics does cause didacticians certain problems of a methodological order. The *didactic* nature of the questions that guide the historical research often means that the ancient texts must be read following a different methodology than that which is usually found in "classical" works and articles concerning the History of Mathematics[5]. Without giving all the credit for a new historiographical invention to the didactical research of mathematics, Thomaidis (1993, p. 71) says:

> "... the questions posed by didactics require new historical research that penetrate to remarkable depths and bring to the surface matters that had not until now occupied the historiography of mathematics".

Recently one has seen many paradigms appear within a didactical historiography framework, each one being a function of a given problem and a particular conception of mathematical knowledge. It is not our purpose here to discuss their similarities or their differences. Yet, we can briefly note that, in our case, our historiographic didactic research program[6] focuses on the investigation of the social roots in which mathematical activity is embedded and in the investigation of the functional triadic dimension of concepts, problems, and the procedures of problem-solving. Given that a concept cannot be limited to its formal verbal formulation, we think that its nature can be better grasped in the dynamic relationships that tie the concept to other concepts, to the problems to which it applies and to the procedures of resolution that one constructs in order to solve these problems (Radford, 1993a; 1993b).

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to no longer use this definition as the starting point and suggest that the concept of limit should be reached through the problems themselves; E. Barbin notes how this change causes the teachers great difficulty.

[5] This point is discussed in detail in Radford, 1995.

[6] The *Didactic Epistemology* whose bases can be found in: Radford, forthcoming$_2$.

The article, which focuses its attention on the specificity of «single unknown algebraic thinking», is part of an on-going research program whose goal is to make a contribution to the understanding of the development of algebraic thinking. Our study derives from the fact that, often, in school programs, the methods of resolution of word-problems used to introduce algebra are based solely on the use of one unknown, while the introduction of the other unknowns follows a few years later[7]. Therefore it is only fitting, from a teaching point of view, to try to understand clearly the characteristics of «single unknown algebraic thought»[8].

In order to obtain some important didactic information related to this, a study of the history of algebraic ideas seems to be one of the most suitable places to explore. In fact, history shows that the invention of the second unknown was a late phenomenon[9]. Thus algebra, for many centuries, was based solely on one unknown. A thorough didactic-epistemological analysis of algebra during its youth - i. e. when it still only had one unknown - can then help us to better understand the profound meaning of the first algebraic ideas and then, further on still, help us to draw out information that can be used in teaching.

Given the above, we propose here a study of mediaeval Italian algebra which will look at various types of problems and the methods used to solve these problems. However, as we said before when we mentioned the main lines of our historical epistemological approach, the comprehension of the cognitive elements underlying the algebraic activity has to take into consideration the socio-cultural dimension by which this activity is embedded and with which it interacts–an interaction that shapes the mathematical activity itself[10]. The cognitive structure of mathematical thought, in general–and of algebraic thought, in particular–has to be scrutinized, in our approach, in its social and intellectual environment and cannot be truly grasped except through the merging of cognitive and social factors. Thus, in the section that follows, we will–within the limitations of this article–make an incursion into the social and intellectual environment of mediaeval Italian algebra.

§ 2 Social and Intellectual Factors in Mediaeval Italian Algebra

The western algebraic current in question here has its roots in Arabic algebra and is part of an intellectual movement dating back to at least the 12th century. It is in this period that Latin translations of certain mathematical works appear in al-Andalus (the region of Spain that was dominated by the Moslems from 711 to 1492). Thus, the first part of the *Traité Concis des règles de l'al-jabar et l'al-muqabala* of the al-Kwharizmi is translated by Robert de Chester in Segovia, in 1145 and by Gerardo of Cremona in Toledo only

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[7] That which is the case in Ontario and Quebec.

[8] In contemporary school programs, algebra is not only introduced through the resolution of word-problems but also, among others, as a tool of generalization and modelization (cf. Bednarz, Kieran, Lee (ed.) forthcoming). Nonetheless, it is necessary to keep in mind that the key concept in the first case is that of the *unknown*, while, in the other cases, the key concept is that of the *variable* and that these two concepts are completely different (cf. Schoenfeld and Arcavi, 1988; for an epistemological analysis of the differences between unknowns and variables, see Radford, forthcoming$_3$).

[9] Historically, there is a gap of more than 13 centuries between the conception of the 'first unknown' and the conception of the 'second unknown': cf. Bednarz *et al.*, 1995.

[10] I do not believe that there exists an ideal mathematical theory –rational, timeless, independent of social stakes– towards which successive mathematical theories would, more or less, move towards over time.

a few years later. Three other important works of this era include: the *Incipit prologus in libro algoarismi di practica arismetrice* of Joannes Hispalensis (John of Spain) that has one chapter about algebra, the *Liber Mensurationum* of Abû Bekr that was written more or less at the same time as al-Kwharizmi's work and translated into latin by Gerardo of Cremona, and the *Liber embadorum* of Savasorda (an Arabic-inspired work belonging to the surveyors' tradition, like that of Abû Bekr's) translated from Hebrew to Latin in 1116 by Plato of Tivoli.

The Western intellectual activity of the 12th century was linked to a favorable reception of the Sciences in royal courts, which encouraged astronomy, agriculture, medicine and mathematics (for instance, the ophthalmologist Sulayman b. Hariṯ al Quti left Toledo and went to Seville in 1160, drawn by the patronage of the almohades); scientific ideas were then spread to other places beyond the bords of the al-Andalus, where a growing economic development of cities like Florence, Venice, and Pisa, needed capable people to efficiently carry out calculations–interest calculations, resale prices, insurance costs for travel (by land or sea), etc.[11] Economic needs led to the rise and the development of new commercial knowledge that resulted in the creation of a new educational institution: the *Botteghe* or the *Scuole d'abaco* (Abacus Schools). These schools were the final step in professional formation for someone who wanted to work in a bank, in public office, or in some commercial office (e.g. cloth manufacturing or construction offices); these schools were also attended by those who, later, wanted to pursue a career in painting, sculpture or architecture (Franci, 1988, p. 184). The *Maestri d'abaco* provided the teaching, which was made up of 'courses'. In one of Florence's schools, directed by Master Francesco Galigai at the beginning of the 16th century, one finds 7 consecutive courses: the first one deals with the basic arithmetical operations of addition, subtraction and multiplication; then there are three courses in division whereby the student learns to divide with one, two and finally three or more digits; next, a course on fractions, another on the rule of three, and finally a course about the Florentine monetary system[12].

Algebra does not seem to have been a part of the 'basic teaching' in the abacus schools. It seems that algebra was only taught to an elite group, reserved for only the few students that had a special interest in mathematics or for those who wished to become abacus masters (cf. Franci, 1988, p. 185; Goldthwaite, 1972-73, p. 426). Nevertheless, it is important to note that even if at the beginning Italian mediaeval algebra appeared to be a tool for the resolution of non-practical problems[13], it then became widely used in commercial applications. According to Master Benedetto of Florence, author of a sort of mathematical encyclopedia of the 15th century[14], it was Master Biagio (deceased circa 1340) who could take credit for

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[11] Cf. Le Goff, 1956.

[12] The preceding information comes from a contract between M° Galigai and Giuliano di Buonaguida della Valle who was hired as an assistant at Galigai's school. The contract was published in Goldthwaite's article, 1972-73.

[13] The most important work that served as a reference point to italian mediaeval algebra –the chapter 15 of*Liber Abaci*, written in 1202 (with a second version written in 1228) by Leonardo Pisano (or Fibonacci), son of a merchant and close to the court of Frederick II– contains only one problem related to commercial mathematics.

[14] Included in the manuscript L.IV. 21 of the *Biblioteca degl'Intronati di Siena* (a description of the content of this important manuscript can be found in Arrighi, 1965).

successfully having applied algebra to the resolution of commercial problems (Franci and Rigatelli, 1988, p. 28).

Besides the potential applicability of algebra in commercial problems (e.g. the calculation of compound interest), the study and the development of algebra were motivated by the prestige and the social recognition given to the *Maestri* (recognition related to the jobs that the abacus master could be called upon to do[15]).

What is known of Italian mediaeval algebra comes from the works of the *Maestri*; works often called *Trattato d'abaco* or *Trattato d'arismetica pratica*, etc. The structure or the chapters of these works vary. In certain cases they are simply a collection of problems with their solutions while, in other instances, the subject matter is presented in a more structured fashion. In this latter case, the steps taken, *grosso modo*, are to introduce the three types of numbers that are "useful in algebra"; i.e. the *radix* (the root) that the Italian abacus masters called *la cosa* (the thing); the *census* (a treasure) which is the square of the thing and, finally, the *denariis* (tokens) or *numerus simples* the numbers having no relationship to the root or square[16]. The combination of these numbers allowed one to obtain the classification of equations into 6 '*cases*'. These *cases* or 'canonical equations' are already outlined in the *Traité Concis sur les règles de l'al-gabr et l'almuqabala* of Al-Khwarizmi written in Bagdad between 813 and 833. What follows are (in modern notation) the six mediaeval *cases*, subdivided in simple and compound (or mixed) equations:

<u>Simple Cases</u>:

(a) $ax^2 = bx$ (b) $ax^2 = c$ (c) $ax = b$

The second case, for example, was stated as: "Treasure equals numbers".

In modern notation, an example of an equation belonging to this case would be $2x^2 = 15$.

<u>Compound Cases (or mixed equations)</u>:

(d) $ax^2 + bx = c$ (e) $ax^2 + c = bx$ (f) $bx + c = ax^2$

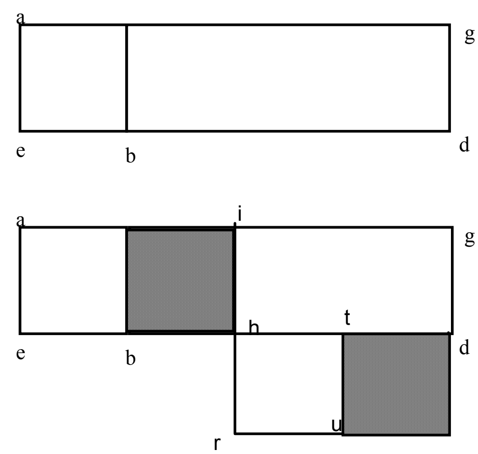
Case (e), for example, was stated as: "Treasures and numbers equal things". For each case, a 'rule' or algorithm was given in order to find the *thing* and the *treasure* (i.e. the square of the thing). Usually, one began by giving the rule for the particular case where $a=1$. Then the case would be reduced to where $a \neq 1$ by dividing the 'coefficients' of the equation by a. For example, in order to solve the case that we have called (e), one begins by giving the 'rule' for the particular equation $x^2 + c = bx$ This rule states that one has to subtract the numbers from the square of half of the things, and that the root of this result has to be subtracted from the half of the things. In modern notations, then, the rule to solve the *case* of $x^2 + c = bx$ is: $x = \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right) - c}$. The "general" *case* (e), $ax^2 + c = bx$ is first solved by dividing the *quantity* of things

[15] Franci (1988, p. 183) mentions that the abacus master would often be called upon to calculate the price for the construction of buildings, to verify or carry out the calculations of mercantile companies and could also be consulted in the calculation of profits and the prices of grains and other merchandises. Goldthwaite (Op. cit. p. 428) mentions that the well-known abacus master, Giovanni di Bartolo, at the beginning of the 15th century, was called in several times as a consultant in the construction of the dome of the *Santa Maria del Fiore Cathedral*.

[16] That is the case of the *Liber Abaci* (Boncompagni, ed., I, 1857, p. 406).

and the *quantity* of simple numbers by the *quantity* of treasures, i.e. by dividing '*b*' and '*c*' by '*a*' in the equation $ax^2 + c = bx$ [17], that which reduces the problem to the *case* $x^2 + c = bx$. Given that negative numbers did not exist at that time, the *Maestri* were aware of the fact that case (e) did not necessarily have a solution[18].

We do not know the exact origin of the rules of the 'people of the al-gabr' (i.e. the algebraists) as they are referred to by Abû Bekr in his *Liber Mensurationum*. However, we do know that the *Traité Concis* of Al-Khwarizmi raises the tradition of algebraists to a scientific level –a tradition, which, perhaps, until that point, had only been handed down orally among surveyors. The *Traité Concis* not only serves as an organized written exposé of the subject matter but it also provides a geometrical explanation (completely different from the Euclidean tradition: cf. Jahnke, 1994, pp. 143-146) of the 'al-gabr's' *rules* (Høyrup, 1994). For example, here is the geometrical explanation for the equation $t^2 + 18\frac{3}{4} = 10t$ (we will encounter this equation in the next section):



The treasure (i.e. term t^2) represents the area of the square *ab* while $18\frac{3}{4}$ is seen as the area of the rectangle *bg*. According to the equation, the square *ab* and the rectangle *bg* form a large rectangle *eg* with an area equal to *10t*. As *ae* = *t*, then, *ag* = *10*. The segment *ag* is divided into two equal parts; with *i* being the middle point. The segment *ih* is lengthened as far as *r* so that *rh* = *bh*. Hence, *rg* is a square with the following area: $\left(\frac{10}{2}\right)^2$, i.e. 25.

Let *t* be the point so that *hi* = *td*. Then, *area rg minus area bg* = *area of rectangle rt*. Therefore, $25 - 18\frac{3}{4} = 6\frac{1}{4}$ = *area of rectangle rt*. Furthermore, it is easy to see that *rt* is, in fact, a square. Therefore its side, *rh*, is equal to $\sqrt{6\frac{1}{4}} = 2\frac{1}{2}$ On the other hand, *ri* = *5*, then *ih* = $5 - 2\frac{1}{2} = 2\frac{1}{2}$, hence the value of the thing.

Certain abacus books (e.g. Pisano's *Liber Abaci* and *La reghola de algebra amuchabale* of Master Benedetto of Florence, Salomone, ed., 1982) provide geometrical explanations of the resolution of algorithms concerning the above-mentioned compound *cases* (i.e. the mixed squares equations). In other books, one can

17 The "coefficients" of mediaeval equations are poorly represented by our modern symbolism: in fact, ax^2, for example, expresses a multiplication between two numbers, *a* and x^2, while in the mediaeval idea it is a matter of expressing a *quantity* of things or treasures that one has. The mediaeval "coefficients" are "numbering numbers" (*nombres nombrants*) (cf. Radford, forthcoming$_4$).

18 Pisano, for example, says that *Case* (e) does not have a solution "unless the numbers are equal or less than the square of half of the roots", i.e. $\left(\frac{b}{2}\right)^2 > c$. Then, supposing that this last condition is met, he adds: "and if the question is not solved through subtraction, it will, without a doubt, be solved through addition", i.e. the thing will be found by making $x = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right) - c}$ (cf. Boncompagni, ed., I, 1857, p. 409). (N. B.: In this article, all the translations from Italian or Latin into English are ours).

find a short introduction of algebraic calculation; it is here that one learns how to carry out elementary operations on binomials. Nevertheless, the heart and the goal of the works or chapters dedicated to algebra is not to explain the geometrical algorithms nor to learn how to carry out calculations on binomials but to show how to use the *techniques* of algebra to solve *word-problems*[19].

Our preceding discussion suggests that algebra was intended to be, above all, a problem-solving tool (based in different *techniques* that we shall analyze in the next section) used to solve a wide range of problems. The question that we can raise now is that of understanding the algebra's problem-solving vocation. It seems to me that the problem-solving nature of mediaeval Italian algebra can be understood, on the one hand, from its conceptual roots with the numerical false-position methods (cf. Radford, forthcoming$_2$) and from the surveyors' geometrical methods that one finds in Abu Bekr's *Liber Mensurationum*; both types of methods are the problem-solving kind as well. However, algebra appears to be a 'research program' with a higher problem-solving fertility than the other 'programs'. Algebra also allows one to tackle different problems using the same technique. In other words, the *family* of problems associated to a given algebraic technique is larger than the family associated to an analogous technique based on numerical or geometrical tools. Algebra appears, then, as a new device to deal with more problems in a more unified and systematic way.

On the other hand, the problem-solving vocation of algebra can also be understood from the social context in which it developed. In fact, the *abacus Master* was, above all, a very practical individual; and not a humanist nor a philosopher. Personal prestige with its social and economical consequences depended upon his individual intellectual capabilities; the resolution of problems and difficult riddles (like the ones that we will see in the 3.2 of the next section) constituted an *ad hoc* instrument of social recognition for the master[20]. The master's algebraic speculations were thus drawn, at the same time, from the applied problems and from non-practical problems–those reserved for the elite of the ‘initiated’ students.

§ 3. Problems and Methods in Italian Mediaeval Algebra: Operating the Unknown

In the previous section, we suggested that mediaeval Italian algebra developed as a set of powerful techniques to solve word-problems. In order to try to understand this technique (each consisting of a family of problems along with the method that solves them), it is worthwhile to examine certain types of problems contained in

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[19] To show the vast extent of the different parts of an algebra text, let's consider *La reghola de algebra amuchabale* of Master Benedetto. This text is divided into three chapters. In the first one, the author introduces the useful numbers in algebra as well as the 6 cases with the geometrical justification of the compound cases, as in the tradition of Al-Khwarizmi. In the second chapter, there is a small introduction to algebraic calculation while the rest is dedicated to the resolution of word-problems. Salomone's edition of M˚ Benedetto tratise contains 104 pages. The first chapter has approximately 18 pages; the second, approximately 10 pages; and the third, that of the word-problems, approximately 74 pages.

[20] For instance, Pisano met at the court of the emperor Frederick II the philospher and mathematician Giovanni da Palermo who proposed to him to solve some mathematical riddles. In a letter sent to the"gloriosissimo principe Federico", included in the first part of his book *The Flos*, Pisano says: " I have started to write a book to the glory of his Majesty and I have called it the Book of Squares" (Picutti, tr., 1983, p. 299). Some of the problems contained in *The Flos* had also the same origin: they were riddles posed at the emperor's court.

the algebra chapters of the abacists' treatises and their relationship with the methods of resolution employed. We are most interested in carefully examining the conceptual bases underlying the operation on the unknown in the problem-solving procedures.

Given the limitations of this article, our study will analyze of two families of problems that appear frequently in the abacist algebra. The problems of the first family (section 3.1) are problems about numbers; i.e. 'theoretical' problems formulated in a mathematical context. Our interest in them lies in the fact that these problems are 'purely algebraic'[21].

The second family of problems (section 3.2) is made up of riddles belonging to traditional non-algebraic mathematics. Unlike the problems of the first family, these problems introduce people, yet are still far from being practical problems. Another difference has to do with the structure of the problem statement.

In section 4, in order to better understand the limits of «one unknown algebraic thinking», we will discuss the scope of algebraic methods based on one unknown.

Given that the Italian algebraists had *rules* to solve the *cases* or canonical-type equations from (a) to (f) (and still others that they later added later, such as $ax^3 = c$; see Egmond, 1978 and also Franci and T. Rigatelli, 1985), our investigation of their problem-solving procedures will be based on the analysis of three structural elements of an algebraic problem-solving procedure. These are: (1) the choice of the unknown and the parametrization[22], (2) the translation process (which makes it possible to obtain an equation that translates the word-problem) and (3) the transformation process (whose goal is that of transforming the translating-equation into one of the 'canonical cases').

§ 3.1 Quasi-equation Problems

The most popular problems solved by algebra in the abacist texts are those whose statement suggests the choice of the unknown and the parametrization at the same time inducing, in a most explicit manner, the setting up of the equation. We shall call these problems the *quasi-equation problems*. In many of these problems, one is asked to divide 10 (some times 12 or another number) into two parts so that, if one carries out certain calculations with these parts, one would obtain a given result. The following problem, taken from Pisano's *Liber Abaci*, is one of the recurring problems in abacist algebra.

"Divide 10 into two parts, add together their squares, and that makes $62\frac{1}{2}$."[23]

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[21] In reality, except for the truly simple problems, these problems cannot be solved by the false-position methods; on the other hand, in light of the historical evidence available today, these problems seemingly can no longer be solved by the surveyors' geometrical methods. Most of these problems appear then as genuine algebraic problems.

[22] Parametrization is the process of finding suitable relationships between the sought-after quantities and the single unknown available, i.e. the thing.

[23] The same problem is found in Raffaello Canacci's *Ragionamenti d'Algebra: i Problemi* (ca. 1490) (Procissi, ed., 1983, p. 28), in Antonio de Mazzinghi's *Trattato di Fioretti* (Arrighi, ed., 1967, p. 23) except that Canacci uses 60 and Mazzinghi uses 82 instead of $62\frac{1}{2}$ and in Piero della Francesca's *Trattato d'Abaco* (Arrighi, ed. 1970, p. 126). (In section 34 we will make reference to Mazzinghi's and della Francesca's problems in a more detailed way). This problem is also found in Rafael Bombelli's Renaissance work, L'*Algebra* (Bortolotti, ed., 1966, p. 341), except Bombelli divides the number 12 into two parts instead of the number 10.

	Comments
Let the first part be one thing, and this multiplied by itself makes a treasure. In the same way, multiply the second part which is 10 minus one thing, by itself; for the multiplication you do this: 10 times 10 equals 100; a subtracted thing multiplied by a subtracted thing makes a treasure to add. And twice 10 multiplied by a subtracted thing makes 20 subtracted things. And so for 10 minus 1 thing multiplied by itself makes 100 and a treasure diminished by 20 things. Adding this to the square of the first part, that is, to the treasure, there will be '100 and two treasures minus twenty things, and this equals 62 ½ denariis. Add, therefore, twenty things to each part, there will be 100 and two treasures equal to 20 things and 62 ½ denariis.	<u>Parametrization and translation processes.</u> The translating equation is, in modern notations: $100 + 2x^2 - 20x = 62\frac{1}{2}$
Throwaway, therefore, 62 ½ from each part, there will remain two treasures and 37 ½ denariis which equal 20 roots; this investigation has thus been brought to third rule of mixed cases, that is, treasures and numbers are equal to roots. In order to introduce the rule, divide the numbers and roots by the number of treasures, that is by 2, and it will make one treasure and 18 ¾ denarii equal to 10 roots.	<u>Transformation process.</u> The translating equation has to be transformed into one of the six conical cases. The transformed equation is: $2x^2 + 37\frac{1}{2} = 20x$
Therefore halve the roots, it comes 5, which multiplied by itself will be 25; from this subtract 18 ¾ , and 6 ¼ remains, subtract the [square] root of these, that is 2 ½, from the half of roots, that is from 5, and it will remain 2 ½; and that is one of the previously mentioned parts; from this right up to 10 there is 7 ½, which is the second. (According to Boncompagni's Liber Abaci edition, I, 1857, p. 411).	<u>Solution process:</u> The equation is solved according to the corresponding case solving-rule.

We need now to discuss in some detail Pisano's problem-solving procedure. With regards to the parametrization process, as illustrated by the quoted text, in order to find each part, Pisano chooses the first number to be the *thing* so that the other part becomes *ten minus the thing*. As we can note, there are no heuristic difficulties in reaching an equation that translates the problem (i.e. $100 + 2t^2 - 20t = 62\frac{1}{2}$). In fact, it suffices to follow the statement of the problem to get the equation; the only difficulties that can arise are the technical computations of the square of the thing and the square of 10 minus one thing. Today, these calculations are carried out according to the "rule of signs"; then perhaps, it would be more appropriately called the "rule of the multiplication of the missing numbers and the added numbers". Once the translating equation has been found, Pisano needs to transform this equation into one of the six canonical cases. The transformations are driven by the key idea of restoring the ‘incompleted’ or ‘broken’ algebraic terms. In order to understand what a ‘broken’ term means, we have to remember that mediaeval mathematics did not have negative numbers. Abacist mathematicians conceptualize algebraic expressions with subtractions as incomplete objects. Thus, the subtracted part (let us say B) in an expression A-B is seen as a missing part of the original part A (this is why the missing parts are often placed at the end of the expressions in the calculations: see Pisano's procedure).

In this line of thought, the first member of the equation $100 + 2t^2 - 20t = 62\frac{1}{2}$ is seen as an incomplete member in that it is *deprived of* or *diminished by* 20 things; according to the algebraic mediaeval idea, this term must be *restored*[24]. In order to accomplish this, Pisano, first allots the 20 missing things to the first member and, then, he allots 20 things to the second member. Next, he subtracts $62\frac{1}{2}$ from each expression and gets $2t^2 + 37\frac{1}{2} = 20t$; this equation, then, can be solved according to the rule of *case* (e). According to the mediaeval tradition, as we said before, the geometrical support is not referred to in this step of the problem-solving procedure; Pisano only shows what calculations have to be done.

To better understand the algebraic problem-solving procedure, we have to note that the sequential structure of the resolution procedure is strongly conditioned by the lack of negative numbers. Thus, Pisano could not have begun by subtracting 100 from each member in the equation $100 + 2t^2 - 20t = 62\frac{1}{2}$ (the member on the right not having enough); nor could he have begun by subtracting $62\frac{1}{2}$ from the member on the left of the equation because, in this case, he would have had to do the same to the member on the right – and the right member should disappear while the new left member should vanish![25] In the *Maestro d'abaco*'s mind an algebraic term cannot be equal to zero. In fact, mediaeval algebraic terms are exactly formed from calculations, and a calculation always gives *something*. An algebraic term is thought of something containing a certain quantity of *numerus simples*. Thus, in the abacist thought, it seems unthinkable that the amount of *numerus simples* carried out by an algebraic term (in this case the term $100 + 2t^2 - 20t - 62\frac{1}{2}$) could be exactly equal to nothing [26].

Concerning the conceptual basis underlying the transformation process, it is also important to note that the rule of *al-gabr*, or of restoration, makes it possible to *operate with the unknown*. In fact, when the

24 The fundamental idea of the rule of al-gabr (from where derives our modern term *algebra*) is actually that of restoring or repairing a incompleted or broken term. It is in this sense that Al-Kwharizmi used it (cf. Radford, forthcoming$_4$) and this is the abacist meaning also. For an etymological study of the term al-gabr see Saliba, 1972.

25 In fact in such a case, we should have $100 + 2t^2 - 20t - 62\frac{1}{2} = 0$

26 In order to better understand why an algebraic term cannot be equal to zero, we must remember that zero was not considered by the abacists as a number (even though the symbol "0" appears in the written form of some numbers, like the number "ten"). In fact, a number was defined as a collection of units (In *De Arithmetica Compendiose Tractata,* his author -Master Guglielmo, 12th Century- says: "Numerus est unitatum collectio vel quantitatis acervus ex unitatibus profusus"). It could be worthwhile to note that the *Maestri d'abaco* needed to relate numbers to something concrete; this can be detected in their recurrent (apparently unnecesary) reference to numbers (*numerus simples*) as denariis. It seems that this concrete way of thinking excluded strongly any possibility of considering zero as a number, then, *a fortiori*, it was impossible to think of zero as a possible solution of an equation or the numerical value of an algebraic term.

'broken' term is repaired, it is necessary to *add* the missing unknown part to the other term of the equation. This action makes it possible to handle (in a particular way) the unknown.

The functioning of this repairing or restoring rule can be stated formally as follows:

> If A(t) and B(t) are two algebraic terms (in the abacist sense) and αt is a certain amount of things, then, from A(t)- αt = B(t) one can get A(t) = B(t) + αt.

However, this rule cannot be seen as a rule of *transposing* terms: *materially*, the new term αt appearing on the right side of the equation is not the same as the corresponding analogous term on the left side. In fact, before being repaired, this last side was missing the term αt. It is not possible to transpose from one side to another side of the equation a term that actually was not there!

The restoration rule makes it also possible to operate on the square of the unknown (the treasure), as it appears in this next problem, taken from Pisano's *Liber Abaci*:

> "I divided 12 into two parts and I multiplied the parts and I divided the result by the difference of the two parts and I got $4\frac{1}{2}$." (Boncompagni, ed., I, 1857, p. 416).

Pisano chooses the smaller part to be the unknown (i. e. the *thing*) so the other part becomes 12 minus the *thing*. Now all he has to do is follow the order of the calculations as indicated in the problem statement. In modern notations, and representing the *thing* by *t*, Pisano's calculations would read as follows[27]:

$$t(12-t) = 12t - t^2, \text{ then } \frac{12t - t^2}{12 - 2t} = 4\frac{1}{2}, \text{ so that } 12t - t^2 = 4\frac{1}{2}\left(12 - 2t\right). \text{Therefore } 12t - t^2 = 54 - 9t$$

By *restoring* the right side (from which 9 things are missing) and the left side (from which a treasure is missing) by giving each part one treasure and 9 things, Pisano arrives at the equation $t^2 + 54 = 21t$ which corresponds to *case* (f) like the equation in the previous example.

Regarding the Italian abacist's algebraic methods, there are, in this problem, two elements to be discussed: the first (occurring in the transformation process) is that of the numerical argument that makes it possible to handle the divisor in the translating equation. In fact, the validity of the algebraic transformation is (implicitly) assured by the property which ties together the three terms of a numerical division (the divisor is equal to the quotient multiplied by the dividend). Transposed to the algebraic realm, this knowledge is not questioned, only accepted. This illustrates a very interesting hierarchical relationship between the existing arithmetical knowledge and the algebraic knowledge under construction.[28]

The second point that we began to discuss in the previous example, namely the *operation on the unknown*. A careful reading of the last problem-solving procedure shows that, in fact, there is an operation *on*

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[27] A commented transcript of the original solution can be found in Radford (1992, p. 61).

[28] There are, however, other cases in which the justification of transformations made on algebraic terms is not based on numerical argument (even if that would have been easier) but on the basis of a geometrical argument (cf. Radford, 1993b, pp. 26-28). A task , which still remains to be accomplished, is that of determining the possible criteria used by the mediaeval Italian algebrists in choosing and inserting some arithmetical and geometrical properties in their algebraic reasonings.

the unknown that differs from the one that derives from the *restoration rule* seen previously. To arrive at the equation $t^2 + 54 = 21t$, from the equation $12t - t^2 = 54 - 9t$, Pisano restores the two sides of the equation, which leads him to the following calculations on the left side of the equation: $12t - t^2 + 9t + t^2 = 21t$. He has had to add 12 things and 9 things to get 21 things, achieving then a new kind of operation *with* the unknown. In fact, this is an operation made within the same side of the equation. It allows the *Maestri* to combine or to put together the unknown terms. We will refer to this rule as the 'combining unknown terms rule'.

The operation on and with the unknown makes it possible within the abacist algebra to transform an equation with broken terms (like $\alpha t - \beta t^2 = \alpha' t^2 + \delta - \beta' t$) into an equation without broken terms (i. e. $(\alpha + \beta')t = (\alpha' + \beta)t^2 + \delta$). The question now is to know if the Abacus Masters were able to transpose an additive unknown term from one side of the equation to the other side (e. g. to transpose the thing "t" in the equation t + 12 = 35t - 60), something that cannot be achieved through the restoration and the "combining unknown terms rules".

Colin and Rojano (1991) detected the operation on the unknown in Bombelli's post-mediaeval work *L'Algebra* (1572). They suggested that the operation on the unknown and more specifically the rule of transposing terms were not explicitly used by mediaeval mathematicians (see also Rojano, 1994, Filloy and Rojano, 1989).

The operation *on* and *with* the unknown (and its square) as well as the transposition rule were, however, quite a widespread systematic practice in Mediaeval Italian (and Arabian) Algebra. For instance, in problem 23 of the *Liber Abaci*, we find, written in modern notations, the following equation: $1040 + 9t^2 - 194t = t^2$; Pisano then adds $194t$ to each side and *takes away* t^2 from each side[29], arriving at the equation $8t^2 + 1040 = 194t$ (Boncompagni, ed., I, 1857, p. 418). In contrast to the 'restoration case', the operation of the unknown additive term t^2 (i.e. the treasure) is done here on the basis of a *subtracting action*. Another example is found in problem 11 of the *Liber Abaci* (Boncompagni, ed., I, 1857, p. 412), where Pisano deals with an equation that, translated into modern symbolism, can be written as follows:

$$\frac{1}{3}\frac{6t}{10-t} + 6t = 39,$$

then, operating with the unknown, he transposes the term $6t$ and gets

$$\frac{\frac{1}{3}6t}{10-t} = 39 - 6t\ .$$

Other examples of this can be found in works by various other Abacus masters. An example of a linear equation ($t + 12 = 35t - 60$) is shown in Canacci's problem in section 3.2 of this article. For further examples see Paolo Gerardi's *Libro di Ragioni* (van Egmond, ed., 1978, problem 1) or *La reghola de algebra amuchabale* of Master Benedetto of Florence (Salomone, ed., 1982, p. 33).

The previous examples show then that the Abacus Master used three different rules to operate the unknown (the restoration rule, the combining unknown terms rule, and the transposition rule). These rules

29 "Restaura ergo res diminutas, et extrahe unum cenum ab utraque parte" –restore, then, the subtracted things and take out one treasure from each part.

allowed them to transform any equation of the form $\pm at^2 \pm bt \pm c = \pm d\,t^2 \pm b't \pm c'$ into one of the six canonical cases (and other cases of higher degree)[30].

In the above-mentioned problems, we were asked to find only two quantities. However, there are many problems in which we are asked to find 3, 4 or even 5 quantities. Here is a problem taken from Raffaello Canacci's *Ragionamenti d'Algebra ...*:

> "Find me three numbers so that the first is to the second as 2 is to 3; and that the second is to the third as 3 is to 4; and that the multiplication of the first and the second multiplied by the third equal the square root of 12". (Procissi, ed., 1983, p. 10)

Here is the first part of the solution:

> Therefore, suppose the first [number] is 2 things, the second is 3 things and the third is 4 things. Multiply 2 things by 3 things; you get 6 treasures. Now you'll say 6 treasures times 4 things equals 24 cubes (chubi) and that must make the root of 12. So, observing the rule, divide the numbers by the cubes and find the cubic root and there you have the value of the thing. Thus, we have the 24 cubes equal to the root of 12 [...]" (Op. cit., p. 10)

At this point Canacci has reached one of the canonical cases that were added to the first six (linear and quadratic) cases.

For our discussion we have to emphasize that even if the problem requires us to find three numbers, with only one available algebraic unknown, the structure of the problem is still chosen in such a way that the parametrization does not pose any problems. The problems, like the preceding one, are chosen *ad hoc*. In fact, as the previous examples suggested, the difficulty of solving quasi-equation problems lies with the calculations to be carried out on a *transformational* level using rhetorical language[31].

§ 3.2 Giving and Receiving Problems

Another family of problems present in abacist algebra concerns two or more people who meet and exchange information about the amount of money they have. The people give each other clues about how much money they have by suggesting that they lend a certain amount to or borrow a certain amount from the others. With these clues, one is supposed to figure out the original amount of money. Here's an example from R. Canacci's *Ragionamenti d'algebra ...*

> "Two men have a certain amount of money. The first says to the second: if you give me 5 denari, I will have 7 times what you have left. The second says to the first: if you give me 7 denari, I will have 5 times what you have left. How much money do they each have?"

Here's the solution:

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[30]Of course, not all the terms in a same side of the equation can be subtracted terms! I am aware of the impossibility of our modern notations to capture the mediaeval ideas. Modern notations are good to carry out modern ideas only.

[31] For instance, it is easy to imagine the difficulties to do calculations in complex quasi-equation problems, like a Pisano's problem that translated into modern notations is the following:. $\left(\frac{t}{10-t}+10\right)\left(\frac{10-t}{t}+10\right)=122\frac{2}{3}$. However, we must not think that the word-problems that we call *quasi-equation problems* are merely equations expressed in rethorical language. In fact, problem statements and equations belong to two entirely different languages. Without this distinction in mind, we risk to misunderstand the abacist algebra.

Canacci's solution:
The first man has 7 things minus 5; the second man has one thing and 5D #.
The second [gives] to the first 5D. He is left with a thing. The first will have 7 things.

Therefore, the first has 7 things minus 5D. He gives 7 to the second who has one thing and 5D, for which he asked, and the first will have 7 things minus 12D. This is equal to 5 times the [amount] of the first. Therefore, multiply the amount of the first by 5 and that gives 5 times 7 things minus 12, that which gives 35 things minus 60D. This is equal to 1 thing plus 12. Even up the parts by adding to each 60D and subtracting a thing from each part. This will give 34 things equal to 72D. Divide the things, as the rule says, and the thing is 2 2/19 *. Therefore, since the thing is 2 2/19 , come back to the beginning of the problem. The first man had 7 things minus 5D, the second man had a thing and 5D. Therefore, the first had 7 things minus 5D, and the thing is 2 2/19 , multiply 7 by 2 2/19 , which gives 14 14/19 , subtract 5; 9 14/19 remains and there you have the amount of the first man. The second had one thing and 5D, add, then, 5 to the thing, that is, to 2 2/19, and you get 7 and 2/19 and there you have the amount of the second man. And you always do your calculations in this manner.
(Procissi, ed., 1983, pp. 38-39)

Comments
first man = $7x - 5$
second man = $x + 5$
After the second person gives the 5 denari, the amounts are $7x$ and x, respectively.

After the first person gives 7 denari, the amounts are $7x - 12$ and $x + 12$, respectively.
Yet, one has:
$x + 12 = 5\,(7x - 12)$
$x + 12 = 35x - 60$
$34x = 72$
$x = 2\ 2/19$

Therefore, the first man = $7x - 5$
the second man = $x + 5$
the first man = $7\,(2\ 2/19) - 5$
$= 9\ 14/19$

the second man = $2\ 2/19 + 5$
$= 7\ 2/19$

\# 5D means 5 denari (money of that time)
* Canacci makes a minor error in his calculation: the thing is actually 2 2/17

In this problem, the purely formal manipulation of the equation (i.e., the syntactical manipulation) is relatively simple if it is compared to the majority of the quasi-equation problems. Its difficulty lies in the role played by the unknown. In this case, contrary to the quasi-equation problems, the unknown does not represent the sought-after quantities thereby not rendering the parametrization as evident. Nevertheless, if the problem is not too different (and if, in particular, the number of people does not increase) the same *pattern* in choosing the unknown and the same method of resolution can solve the problem. That is the case of the next problem of Canacci's *Ragionamenti d'algebra ...* that follows the problem discussed above; it is stated as follows:

> "One man says to another: if you give me 5 denari, and I add them to what I have, I will have 1 and 7 times what you have left. The second says to the first: if you give me 7 denari and I add them to what I have, I will have 5 times what you have minus two." (Op. cit., p. 39).

However, if the number of people increases, the problem becomes very difficult to solve using only one unknown; Cardano himself says as much in his *Ars Magna* (Witmer, ed., 1968, p. 71).

It should also be noted that the «giving and receiving problems» seem to belong to a non-algebraic tradition of riddles[32]. For a long time, they were solved by false position methods. Using algebra to solve these problems in mediaeval times (as well as others of the same type that cannot be solved by false position methods)[33] the Masters and their pupils had the opportunity to enjoy and to prove to themselves the fertility and the superiority of algebra with regards to arithmetic.

§ 4. On the scope of algebraic methods

We said before that one of the most frequent kinds of problems in abacist texts is that in which one is asked to divide 10 into two parts so that, if one carries out certain calculations with these parts, one would obtain a given result. In section 3.1 we saw that one parametrization used by the *Maestri* consisted in expressing the parts or sought-after numbers as "*a thing*" and "*k minus a thing*" ("k" being the number to be divided into two parts). However, there was another parametrization based on ‘the half of the number’: the sought-after numbers are expressed as "$\frac{k}{2}$ *plus a thing*" and "$\frac{k}{2}$ *minus a thing*". The origin of the last parametrization seems to date back to the emergence of algebra: in fact, this parametrization would be at the root of the "algebraic version" of a Babylonian false position method[34]. Let's look at a few abacist examples of this kind of parametrization. In a problem from the *Liber Abaci*, (problem 70 in Salomone's translation, 1984, pp. 71-73), Pisano is drawn to divide 10 into two parts such that the sum of the first divided by the second and the second divided by the first gives 3. It is also a classic problem of abacist algebra. In fact, problems 63-71 of the *Liber Abaci*, according to the manuscript L.IV.21 of the *Biblioteca Comunale di Siena*, are related to this type of problem. To solve problem 70, Pisano designates the first part as 5 plus a thing and the second as 5 minus a thing.

In contemporary notation, the equation reads:

$$\frac{5+x}{5-x}+\frac{5-x}{5+x}=3$$

Pisano multiplies 5 minus a thing by 5 plus a thing which equals 25 minus a square; he then multiplies this result by 3 and gets 75 minus 3 squares. He equates this to the sum of the square of 5 plus a thing and to the square of 5 minus a thing, i.e. 50 plus two squares. Once he has arrived at this equation, which we will write as $75-3x^2=50+2x^2$ he *restores* the left side of the equation. In order to do this, he gives this side the 3 missing squares and adds 3 other squares to the right side. In modern notation, the result would be $75=50+5x^2$ He then subtracts 50 from each expression and gets $25=5x^2$ where he finds that the thing is equal to the root of 5. He can then easily deduce the sought-after numbers.

Here is another example taken from Maestro Antonio de Mazzinghi's *Trattato di Fioretti*.

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[32] In reality, these problems are ancient: they can be found in a collection of problems (probably gathered at the beginning of the 6th Century A. D.) called the *Greek Anthology* (see Paton, ed., 1979, Book XIV, problems 145 and 146). Nevertheless, one of Plato's commentaries suggests that these problems date back to, at least, the 5th Century B. C.

[33] For a beautiful example, see Giovanni di Bartolo's *Certi Chasi* (Pancanti, ed., 1982, problem 10, pp. 18-21).

[34] See Radford, 1993a; however, the idea is much more precise in Radford, forthcoming3.

"Divide 10 into two parts so that the sum of their squares equals 82."

Mazzinghi provides three different solutions to this problem, however, it is the first one in which we are interested. The parametrization is the following:

"Make it so that the first part is 5 plus a thing and the second part is 5 minus a thing". (Arrighi, ed. 1967, p. 23)

Another example of this type of parametrization is found in problem 20 of Bombelli's *L'Algebra* (Bortolotti, ed., 1966, p. 326).

This parametrization disappears progressively during the Renaissance as a result of a standardization of parametrization strategies. In fact, the first parametrization mentioned at the beginning of this section (let us call it the ‘direct parametrization’) tended to replace the parametrization based on ‘the half of the number’.

The primacy of the ‘direct parametrization’ over the ‘the half of the number’ parametrization can already be detected in Piero della Francesca's *Trattato d'Abaco* (15^{th} Century; ed. Arrighi, 1970). In fact, della Francesca solves many problems of the type ‘Divide 10 into two parts such that ...’. If we denote by *a* and *b* the parts into which 10 has to be divided, some of these problems in the della Francesca's *Trattato d'Abaco* are the following (*Op. cit*. pp. 126 - 129):

(1) ab =21[35] (2) $a^2 + b^2 = 58$

(3) $\frac{a}{b} + \frac{b}{a} = 4\frac{1}{4}$ (4) $\frac{10}{a} + \frac{10}{b} = 10$

(5) $ab = 5\frac{1}{4}(a-b)$ (6) $a^2 + b^2 + (a-b) = 54$

(7) $\frac{ab}{a-b} = 12$ (8) $ab = (a-b)^2$

In each problem, one of the sought-after numbers is represented by the *thing* and the other is represented by 10 less the *thing*. Another example is given by Bombelli's *L'algebra*: even though he sometimes uses ‘the half of the number’ parametrization, the ‘direct parametrization’ is used quite more frequently.

However, the ‘direct parametrization’ as well as ‘the half of the number’ parametrization cannot be applied generally to problems dealing with more than two sought-after quantities. How, then, can one face problems with more than two sought-after quantities using only one unknown? We have seen in the previous section that one way was to make the sought-after quantities related in a certain specified proportionality (in fact this was the most frequent issue; see Cannaci's example at the end of section 3.1). The next example (where della Francesca asks to divide 10 into three parts) does not use a proportional relationship between the sough-after parts. Its interest comes from the fact that della Francesca keeps the direct parametrization that he used *successfully* in the previous problems (where there were two sought-after quantities only). The parametrization does not solve the new problem in general terms (a problem which has in fact more than one solution). Furthermore, as far as we can see from the text, della Francesca gives a rather arbitrary value to one of the sought-after quantities (as we have observed some students doing when they face problems with more

[35]The whole problem is then: "Divide 10 into two parts such that their multiplication makes 21".

than one unknown[36]) and seems to believe that he has solved the problem completely. The problem (*op. cit.* p. 134), translated into modern symbols, is the following:

$$a+b+c=10$$

$$ac=b^2$$

The problem-solving procedure begins in these terms[37]:

"Say that the first part is a thing, the second part 5 minus $\bar{I}$ thing and the third 5. Multiply $\bar{I}$ thing by 5, you get $\bar{5}$ things, and 5 minus $\bar{I}$ by 5 minus $\bar{I}$ makes $\overset{\square}{\mathrm{I}}$ and 25 minus $\overline{10}$ things".

At this point, della Francesca has reached the translating equation: $5x = x^2 + 25 - 10x$. The next step is to transform the equation into one of the 6 canonical cases. He says: "Restore the parts giving to each part $\overline{10}$ things". The final equation is then $15x = x^2 + 25$; after solving this equation through the corresponding rule, he finds that the first sought-after part is $7\frac{1}{2} - \sqrt{31\frac{1}{4}}$. He then says: "And the second [sought-after part] is 5 minus what is left from $7\frac{1}{2}$ minus $\sqrt{31\frac{1}{4}}$; and the third [sought-after part] is 5."[38]

This marvelous example shows us how a specific method (applied successfully to some specific problems) is applied to more complex problems. Whether or not della Francesca was aware of his incomplete solution, the example shows a concrete limitation of the particular methods of the «one unknown algebra». It was through the invention of the other unknowns that radical changes took place among the methods and the parametrization strategies. Thanks to the many possibilities offered by the emergence of these other unknowns, later on, ancient mathematicians were able to interpret the problem statement right away. Then again, one will also see other difficulties arise[39].

§5. Some implications for teaching

The didactic epistemological analysis of abacist algebra made here shows that algebra was considered as a *tool* or a *technique* to solve riddles and word-problems. Our discussion about the social and intellectual context of the Italian mathematical activity of the 13th - 15th centuries, suggests that the problem-solving tool status of algebra of the time was intricately rooted in social and economic elements which shaped the practical nature of the *Maestri d'abaco*'s knowledge (e.g. the social prestige that they could enjoy in their community, the possibilities that such a knowledge brought them in their work as mathematics teachers or as consultants to private and public businesses). The social elements cannot however give a complete account of abacist algebra. There are also cognitive elements to be considered. Nevertheless, the cognitive elements cannot be understood by isolating them from their own ‘intellectual mathematical context’, which shed some

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36 See Radford, 1994.

37 Della Francesca denotes the thing by $\bar{I}$ and the treasure (censo) by $\overset{\square}{\mathrm{I}}$.

38 The problem ends with an alternative way to compute the second sought-after part: given that a= $7\frac{1}{2} - \sqrt{31\frac{1}{4}}$ and c=5, b can be deduced from the condition $ac = b^2$.

39 I deal with this subject in: *Sur l'invention d'une idée mathématique: la deuxième inconnue* (manuscript in progress).

light on the objectual and conceptual organization of the mathematical content itself. This is why we needed to consider the Arabian algebra legacy and the occidental mathematical activity of the 12th century.

Seeing the cognitive elements of the abacist algebra within the tradition of the surveyors and of the Arabian algebra, makes it possible to understand the core of the abacist algebraic problem-solving procedures and the very fundamental ideas underlying the operation of the unknown. Concerning this last point, the operation of the unknown -as we have seen- was based mainly on three basic rules: the first one is rooted in a very particular idea, which consists of seeing an algebraic expression, like 54-9t, as a defective or broken expression. To 'repair' it, we need to *restore* the missing part, that is 9t. This is done by applying the Arabian rule of *al-gabr*. The second rule which makes it possible to operate with the unknown, is that of combining the unknown terms within a same side of the equation (e.g. $12t - t^2 + 9t + t^2 = 21t$). The third rule is the transposition rule (e.g. in the Pisano's example, the term 6t is transposed:

$\frac{1}{3}\frac{6t}{10-t} + 6t = 39, then\ \frac{1}{3}\frac{6t}{10-t} = 39 - 6t$).

On the other hand, our analysis shows how the lack of negative numbers in abacist algebra shaped the structure of the algebraic problem-solving procedures (see problem 1, section 3.1). Furthermore, it should be noted that the lack of negative numbers led the abacist to conceptualize subtraction in a particular way, which makes subtraction play an unsymmetrical role to that of addition. For instance, in abacist algebra, an expression like 54+9t does not need to be repaired, through to the homologous expression 54-9t does.

The particular algebraic conceptualization of the addition and subtraction operations, led the abacist to see the terms in the equation $2x^2 + 54 = 21x$ in their 'natural state'; they could not have assimilated this equation to one of the form $ax^2 = bx + c$ not because they did not know rule for transposition terms -something they effectively knew, as we saw previously- nor because of the limitations of the rhetorical algebraic language that they used, but because the equation $2x^2 = 21x - 54$ has lost, in their mathematical conceptualization, its 'natural equilibrium'.

We can see then that what led the abacist to distinguish between the equations of the form $ax^2 = bx + c$ and those of the form $ax^2 + c = bx$ was -apart from the weight of the Arabian algebraic tradition- the specific conceptualization of algebraic subtraction. It could be worthwhile to note at this point the abacist algebrists were not concerned with the problem of the 'unification' of the six canonical cases, particularly the 3 mixed cases (the cases (d), (e) and (f): see section 2) into a single case. It was a post-mediaeval problem[40]. We can hypothesize that the 'unification´ problem became a genuine mathematical problem only when negative numbers had developed to occupy a minimal mathematical conceptual space.

[40]The abacist were rather interested in producing new cases. A good example is given by a treatise called *Aliabraa argibra*, attributed to M° Dardi of Pisa (*fl.* 1344): in this treatise we find 198 different type of equations!

Let us turn now to the teaching of algebra. In section 1, we said that the history of mathematics may give us a new perspective on teaching. Of course, we are not saying that our students have to follow the same path as that of ancient mathematicians. Rather, it is a question of better understanding the nature of mathematical knowledge and to find, within its historical structure, novel teaching possibilities. One of the points concerning the curriculum that can be raised is that of the links between algebra and negative numbers. To my knowledge, these two subjects are usually taught independently. History may suggest some new links (e.g. integrating the concept of negative numbers in an algebraic teaching sequence).

Another point that deserves to be discussed is that of the meaning of algebra in an introductory course. Abacist algebra appears -we were able to observe- as a method for solving problems. Abacist algebraic knowledge evolved mainly at the "between problems" level. Each family of problems poses different difficulties: certain difficulties appear at the parametrization level; others at the syntactic level, etc. Furthermore, the algebraic language evolved from a problem solving tool to a mathematical object. (The peculiar symbolic notations used by della Francesca –where the geometrical meaning is quite obvious– is just one example of a sequence of efforts made by Mediaeval and Renaissance mathematicians to handle the unknown and its powers in a more comfortable way than that provided by rhetorical language[41]). Instead of presenting algebra as an achieved complex language, will our students gain a better understanding of algebraic concepts if we present them algebra as a problem-solving tool for facing some specific family of problems, making that the symbolic language emerges and evolves from the problem-solving activity itself?

The didactic way in which the possible links mentioned above could be done remains to be discussed. However, even though we decide not to take into account the historical-epistemological insights, I claim that our knowledge about the meaning of algebraic thinking will have matured. For instance, it appears to me that once we have seen the role played by the absence of negative numbers within the structure of abacist mediaeval algebraic problem-solving procedures, it is very hard to see their *presence* in modern mathematical curricula in the way that we saw them before.

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[41] Another effort can be found in Bombelli's symbolic language (for a study of the Bombelli's symbolic language see Colin and Rojano, 1991).

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